IMPLICATIONS IN THE COHOMOLOGY OF H-SPACES

BY

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0. Introduction

0.1. Summary of results. In his investigations of the cohomology of H-spaces, W. Browder introduced the notion of implications in Hopf algebras (see e.g. [3, p. 357]). In this study we show that in some important cases an implication in the cohomology of H-spaces (if it is not a p^{th} power) takes the form of a (nonstable) secondary cohomology operation. Some properties of this operation were investigated in [10]. The main calculations are carried out in Section 2 where we study the operation ϕ, defined on $(\ker \xi)^{2m}$ ($\xi x = x^p$) and associated with the inequality $e(\beta p^m) > 2m$ (e is the excess).

The most significant results of this study could be summarized as follows:

THEOREM A (Corollary 1.7). *Let (X, μ) be an H-space and $B \subset H^*(X, Z_p)$ a sub-Hopf algebra closed under the action of the Steenrod algebra $\mathfrak{A}(p)$; then there exists an H-space (G, μ_0) and an H-mapping $f : (X, \mu) \to (G, \mu_0)$ so that $\operatorname{im} f^* = B$. If B is associative G may be assumed homotopy associative.*

Moreover, G can be taken to be a product of Eilenberg-McLane spaces with an exotic multiplication.

Next, consider the fiber $\tilde{X}, j : \tilde{X} \to X$, of the map $f : X \to G$. Since G is a product of Eilenberg-MacLane spaces

$$\operatorname{im}(j^*) \cong H^*(X, Z_p)/\!\!/B \quad \text{in } H^*(\tilde{X}, Z_p).$$

In particular, if $\bar{\mu}^*(x) \in B \otimes B$ then $j^*(x)$ is primitive and we have

THEOREM B (Corollary 2.2). *Let p be an odd prime, $B_0 \subset \ker \xi$ (ξ being the p^{th} power operation) and $x \in \ker \xi$ satisfies $\bar{\mu}^* x \in B \otimes B$ and suppose that (X, μ) is homotopy associative and $H^*(X, Z_p)/\!\!/B$ is cocommutative. There exists a secondary operation ϕ defined on $\ker \xi$ so that for every $z \in (H^*(X, Z_p)/\!\!/B)^*$ we have*

$$\langle x, z \rangle = \langle \phi(x), z^p \rangle.$$

Remark. This is an implication theorem in the sense of W. Browder (see [3, p. 357]) and it considerably strengthens his results.

As a typical application of Theorem B we have

THEOREM C (Theorem 3.1 (d)–(e)). *If X is a homotopy associative H-space, p an odd prime, and $H^*(X, Z_p)$ is primitively generated, then $H^*(X, Z_p)$ is a free algebra (i.e., a tensor product of an exterior algebra on odd-dimensional generators and a polynomial algebra on even dimensional generators).*

Received April 21, 1969.