RELATIONS IN CATEGORIES

BY

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Introduction

This paper is concerned with relations in general categories. MacLane [1], Puppe [2], Hilton [3] considered the abelian case; namely, the categories of relations extending abelian categories. In order to develop a general theory which still includes the classical case, we need some structure to insure the existence of a good factorization of morphisms; it seems that the bicategory structure in the sense of Isbell [4], Semadeni [5] is adequate. By imposing this structure, we include the classical counterpart in the form of categories with images. Moreover, since a category can generally be made into a bicategory in several ways, the choice may be of importance. To obtain a reasonable system it is likely that only few conditions on the chosen type of category (finitely complete [6] bicategories) may be relaxed.

For completeness some facts about bicategories are established. Relations are introduced and composition defined using the set-theoretical relations as a natural model. Associativity is shown to be false in the general case. The associative case is characterized by a categorical form of the Ore conditions in semigroups and rings. However, in the general exposition associativity is not assumed and it seems that even the nonassociative case can be handled for some purposes. Functors and extensions are considered. In the last section congruences are introduced and a regularity property is proved for congruences with respect to group-like structures.

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We denote the class of morphisms $A \to B$ in $\mathcal{C}$ by $\mathcal{C}(A, B)$. $|\mathcal{C}|$ denotes the class of objects of $\mathcal{C}$. (Co-) Retractions are (left-) right-invertible morphisms. A product of A, B in $\mathcal{C}$ is usually denoted $(A \times B, \pi_A, \pi_B)$ and the unique morphism ζ into $A \times B$ such that $\pi_A \zeta = \xi$, $\pi_B \zeta = \eta$ is denoted by $\{\xi, \eta\}$. We denote a pullback

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "v"'] & \cdot \arrow[d, "\xi"] \\ \cdot \arrow[r, "\eta"'] & \cdot \end{tikzcd}$$

by $\downarrow v, u, \eta, \xi \downarrow$.

(0.1) A *bicategory* is a category $\mathcal{C}$ with a structure consisting of two subcategories $\mathcal{I}$ and $\mathcal{S}$ such that the elements of $\mathcal{I}$ are monics, those of $\mathcal{S}$ are epics;

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