A REMARK ON THE BIRKHOFF ERGODIC THEOREM

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In this note we will prove the following theorem:

THEOREM. *Let T be a 1–1, invertable, measure-preserving, ergodic transformation of a measure space X onto itself. Let*

$$f^*(x) = \sup_n (1/n) \sum_{i=1}^n f(T^i(x)).$$

(a) *Assume X has finite measure. Then for $f \geq 0$, $f^*(x)$ is integrable if and only if $[f(x) \log (x)]^+$ is integrable (g^+ is the positive part of g).*

(b) *Assume Z has infinite measure. Then for $f \geq 0$, $f^*(x)$ is not integrable.*

The "if" part of (a) is well known and is only stated here for the sake of completeness.

This paper has as its starting point the following theorem of Burkholder: Let X_i be a sequence of independent identically distributed, non-negative random variables. Then $\sup_n (1/n) \sum_{i=1}^n X_i(\omega)$ is integrable if and *only if* $[X_i(\omega) \log (X_i(\omega))]^+$ has finite expectation. Gundy, in an unpublished paper, proves a reverse maximal inequality from which he deduces the above theorem. (This is generalized in Proposition 1.) Gundy also suggested that his theorem holds in the more general case of an ergodic transformation, and that is what we prove here. This seems to be the natural setting for the theorem, since it does not hold for the identity transformation, $T(x) = x$. Furthermore, the theorem does not seem to generalize in a natural way to the operator case, since it does not hold for the linear operator that sends every function into a constant.

LEMMA 1. *Given a set D, of non-zero measure, we can find disjoint sets A_i^j, $1 \leq i < M_j < \infty$, $1 \leq j < \infty$, such that*

$$T(A_i^j) = A_{i+1}^j, \quad \textit{unless } i = M_j - 1,$$

$$\bigcup_{j=1}^\infty A_1^j = D \quad \textit{and} \quad \bigcup_{j=1}^\infty \bigcup_{i=1}^{M_j-1} A_i^j = X.$$

Proof. For each $x \in D$ let $N(x)$ be the first integer ≥ 1 such that $T^{N(x)}(x) \in D$. Let A_1^j be the set of x, in D, such that $N(x) = j$. Let $M_j = j$ and let $A_i^j = T^{i-1}(A_1^j)$ for $i \leq j$. The A_i^j are disjoint because T is 1–1 and their union is X because T is ergodic.

PROPOSITION 1. *Fix $\alpha > 0$. Let E be the set where $f^* \geq \alpha$. Let F be the set where $f \geq \alpha$. Assume that $m(X - E) \neq 0$ ($m(C)$ is the measure of C). Then $\alpha \cdot m(E) \geq \frac{1}{2} \int_F f$.*

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