# A CHARACTERIZATION OF MONOTONE FUNCTIONS

BY

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The purpose of this note is to prove the following theorem:

THEOREM. *Let $f(x)$ be a real-valued function of a real variable satisfying the following.*

(a) *$f(x)$ is approximately continuous, i.e., for each $x_0$ and $\varepsilon > 0$ the set of $x$ such that $|f(x_0) - f(x)| < \varepsilon$ has density 1 at $x_0$;*

(b) *For each $x_0$, let $E$ be the set of $x$, such that $f(x) - f(x_0) \geq 0$. Then*

$$\limsup_{|h|\to 0} m[E \cap (x_0, x_0 + |h|)]/|h| \neq 0$$

*where $m(C)$ is Lebesgue measure of $C$.*
*Then $f(x)$ is monotone increasing and continuous.*

One may be tempted to weaken (b) as follows: (b′) for each $x_0$ the set of $x$ such that $(f(x) - f(x_0))/(x - x_0) \geq 0$ does not have 0 density at $x_0$. In this case, however, the conclusion is false, even if we assume $f(x)$ to be continuous. (We will describe such an example at the end of this note.)

Condition (b) may be replaced by the following weaker condition:

$$\limsup_{x\to x_0} (f(x) - f(x_0))/(x - x_0) \geqq 0, \qquad x > x_0$$

neglecting any set of values of $x$ that has density 0 at $x_0$. This follows from our theorem because if $f(x)$ were not monotone we could add a linear function with positive slope to $f(x)$ in such a way that the result is still not monotone but condition (b) is satisfied.

Without loss of generality we will assume $f(x)$ to be defined only on the unit interval. We will now prove Theorem 1.

LEMMA. *Let $A$ be a measurable set in the unit interval, $I$, of measure $\gamma > 0$, and $r$ a real number $> 1$. Assume that $2r\gamma < 1$. Let $U$ be the union of all the intervals $J$ in $I$ such that $m(A \cap J)/m(J) > r\gamma$. Then $m(U) < 2/r$.*

*Proof.* Pick a finite subset $S$ of the intervals which make up $U$, such that the measure of their union is within $\varepsilon$ of the measure of $U$.

If there is an interval in $S$ which is contained in the union of the remaining intervals delete it from $S$. Call the new collection $S_1$. Delete from $S_1$ an interval (if there is any) that is in the union of the remaining interval in $S_1$. Call the result $S_2$. We will eventually get a collection $S'$ so that no interval in $S'$ is in the union of the remaining intervals and the union of the intervals of $S'$ = the union of the intervals of $S$. It is easy to see that no point is in

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Received September 22, 1968.

[1] This research is supported in part by a National Science Foundation grant.