EX-HOMOTOPY THEORY I

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This is the first of a series of studies of a generalization of ordinary homotopy theory. The basic notion is simple enough and seems to have occurred more or less simultaneously to others as well as myself. I understand that Heller and Hodgkin, independently, have given applications to the Eilenberg-Moore spectral sequence. Also McLendon [4] has announced a generalization of the Adams spectral sequence on these lines. My own work is directed towards applications of a different type. Some of these are contained in [3], to which the present note is closely related. Others will be given subsequently.

1. Basic notions

Let B be a space. By an *ex-space* (over B) we mean a triple (X, σ, ρ), where X is a space and

$$B \xrightarrow{\sigma} X \xrightarrow{\rho} B$$

are maps such that $\rho\sigma = 1$. Normally it will be sufficient to denote the ex-space by X. We refer to σ as the *section*, to ρ as the *projection*. Together they constitute an *ex-structure* on the *total space* X over the *base space* B. Notice that B can always be regarded as an ex-space over itself, with $\rho = 1 = \sigma$. We refer to this as the *trivial ex-space* over the given base.

We describe an ex-space X as *proper* if σB is a closed subspace of X. When this condition is satisfied we can embed B in X, by means of σ, so that ρ constitutes a retraction. Instead of regarding B as a retract of X we regard X as an "extract" of B. This change of view opens up the prospect of the following development.

We shall outline a theory which reduces to ordinary homotopy theory when B is a point. The generalization proceeds on formal lines, for the most part; whenever we meet a basepoint, in the ordinary theory, we replace it by B, using the section and projection in an appropriate way.

Starting from the given base space we have begun to construct a new category out of the category of topological spaces. The objects in the new category are ex-spaces. We now define the morphisms. Let X_i $(i = 0, 1)$ be an ex-space over B with section σ_i and projection ρ_i. By an *ex-map* $f : X_0 \to X_1$ we mean an ordinary map such that

$$f\sigma_0 = \sigma_1, \qquad \rho_1 f = \rho_0, \tag{1.1}$$

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