SUBORDINATE FAMILIES OF ANALYTIC FUNCTIONS[1]

BY
LOUIS BRICKMAN

We shall be concerned with one-parameter families of analytic functions $F(z, t)$, $0 \leq t \leq T$. More specifically, we are interested in conditions for $F(z, t)$ to be subordinate (definition below) to $F(z, 0)$ in $|z| < 1$ for each fixed t. The idea of subordination of a whole family of functions is of frequent occurrence in geometric function theory. In fact we take the view that this idea unifies the theory. For example, if $f(z)$ is univalent in $|z| < 1$ and $f(0) = 0$, then $f(z)$ is starlike if and only if $F(z, t) = (1 - t)f(z)$ is subordinate to $F(z, 0)$ for each t satisfying $0 \leq t \leq 1$. (Throughout the paper, as in this example, we shall have $F(z, 0) \equiv f(z)$.)

The origins of our technique are in the paper [2] of M. S. Robertson, and our Theorem 1, which provides a necessary condition for subordination, is a reformulation of Robertson's theorem. However we are able to replace the original assumption that $f(z)$ is univalent by the simple requirement $f'(0) \neq 0$. One benefit of this modification is the acquisition (after Theorem 2) of a whole class of theorems of the form "Subordination implies univalence and convexity". A member of this class of theorems is the recent result of T. H. MacGregor [1] in which

$$F(z, t) = \frac{1}{t}\int_0^t f(ze^{i\theta})\, d\theta.$$

The four theorems presented here can be described briefly and fairly accurately with the terminology of Differential Calculus. Theorem 1 provides a "first-derivative" criterion necessary for subordination. However if the "first derivative" "vanishes" it is possible and useful to make an assertion concerning the "second derivative"; hence Theorem 2. Theorems 3 and 4 are converses of Theorems 1 and 2 respectively. Thus they provide sufficient conditions for subordination. Theorems "of order greater than 2" can be stated, but it is not clear that they would be useful.

The basic definition of subordination is as follows. If $f(z)$ and $g(z)$ are analytic in $|z| < r$, we say that $g(z)$ is subordinate to $f(z)$ in $|z| < r$ if $g(z) = f(\omega(z))$ for some "Schwarz function" $\omega(z)$. That is, $\omega(z)$ is analytic in $|z| < r$ and $|\omega(z)| \leq |z|$. We shall write $g(z) \prec f(z)$ to express this relationship. A commonly occurring set of conditions sufficient for subordination is that $f(z)$ be univalent, that range $g \subset$ range f, and that $g(0) = f(0)$.

We wish to thank Professor MacGregor for calling our attention to [2] and for several helpful conversations.

Received November 15, 1968.
[1] Research supported by a National Science Foundation grant.