ISOMETRIC IMMERSIONS INTO SPACES OF CONSTANT CURVATURE

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1. Introduction

In this paper we first give a unified account of invariant tensor methods in Riemannian geometry and their application to the study of isometric immersions of Riemannian manifolds. The ideas presented here are more or less implicit in classical work, but the details are usually erroneously oversimplified or treated incorrectly in the more recent literature. For this reason, we give our treatment in some detail.

We first discuss, in Section 2, covariant differentiation of tensor fields over mappings, proving the generalized structural equations in detail. This material is indispensable to our treatment of the immersion theory in Section 3, where we develop analogues of the classical Gauss, Codazzi, and Ricci equations. Our version of these equations is, on the one hand, completely index-free, so the roles of the various operators (connection, normal connection, second fundamental form, etc.) in describing the geometry of the immersion is made clear. At the same time, these equations are valid for vector fields over arbitrary mappings; thus they retain all the flexibility of the classical equations.

We then use these equations to prove the following theorem, pausing to note the most general conditions needed at each stage of the proof.

THEOREM. *Let $I : M^d \rightarrow \bar{M}^{d+k}$ be an isometric immersion of a complete d-dimensional Riemannian manifold M^d in a $(d + k)$-dimensional Riemannian manifold $\bar{M}^{d+k}$ of constant curvature K. Then there exists a complete n-dimensional totally geodesic submanifold L of M which has constant curvature K in the induced metric, and which is totally geodesically immersed in $\bar{M}$ by I. Here n is the minimal value of the index of relative nullity.*

This theorem generalizes a result of B. O'Neill and E. Stiel [7], where both M and $\bar{M}$ have constant curvature K. P. Hartman [5, Lemma 3.1 (v)] proved our theorem for the $K = 0$ case, generalizing the previous result of O'Neill [6] for M and $\bar{M}$ flat (see also S. B. Alexander [1]). The index of relative nullity ν was defined by Chern and Kuiper [3], who also showed, essentially, that if M and $\bar{M}$ both have constant curvature K, then $\nu \geq d - k$, so the theorem is not vacuous for $d - k$ positive.

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