PROJECTIVE DIAGRAMS OF INTERLOCKING SEQUENCES

BY

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1. Introduction

A complex over an abelian category $\mathcal{A}$ with enough projectives is projective as an object in the category of complexes precisely when the kernel of each of its (boundary) maps is projective and the homology of the complex is zero in each dimension. This paper shows that the projective objects in a more general category of diagrams of interlocking sequences over $\mathcal{A}$ are those diagrams with the kernel of each map in the diagram projective and with each of the sequences exact.

Let N be a fixed integer greater than 2. Denote by $\mathcal{R}$ the pointed category whose objects are pairs (s, t) of integers satisfying $s - N < t < s$, whose hom-sets are given by

$$\begin{aligned}\mathcal{R}((s, t), (u, v)) &\cong Z_2 \quad \text{when } u - N < t \leq v < s \leq u \\ &\cong \{0\} \quad \text{otherwise,}\end{aligned}$$

and whose composition

$$\mathcal{R}((u, v), (w, x)) \otimes \mathcal{R}((s, t), (u, v)) \to \mathcal{R}((s, t), (w, x))$$

is the isomorphism $Z_2 \otimes Z_2 \to Z_2$ when all three hom-sets are Z_2, and zero otherwise. (Here $\otimes$ denotes the coproduct of pointed sets.)

For $(s, t), (u, v) \in \mathcal{R}$, $t \leq v$, $s \leq u$, the symbol $(s, t; u, v)$ will denote the non-zero element of $\mathcal{R}((s, t), (u, v))$ when it has one, and the zero element otherwise; the rule of composition in $\mathcal{R}$ may be expressed by

$$(u, v; w, x) \cdot (s, t; u, v) = (s, t; w, x).$$

Put $\delta = (s, t; u, v)$. The integer $l_\delta = u - s + v - t$ is called the *length* of δ. If $l_\delta \geq N - 1$ then δ is zero. If $l_\delta = N - 2$ then δ is non-zero precisely when $u = N + t - 1$, $v = s - 1$. If $l_\delta = 0$ then $u = s$, $v = t$. If $l_\delta = 1$ then δ is non-zero and either $u = s + 1$, $v = t$ or $u = s$, $v = t + 1$. If δ is non-zero (i.e. $u - N < t \leq v < s \leq u$) then we also define integers $m_\delta = s - v - 1$ and $n_\delta = N - l_\delta - 2$. Notice $0 \leq m_\delta \leq n_\delta \leq N - 2$.

An $\mathcal{R}$-*sequence* is a diagram in $\mathcal{R}$ of the form

$$(u, v) \xrightarrow{(u,\, v;\, t,\, v)} (t, v) \xrightarrow{(t,\, v;\, t,\, u)} (t, u)$$

or

$$(t, v) \xrightarrow{(t,\, v;\, t,\, u)} (t, u) \xrightarrow{(t,\, u;\, v + N,\, u)} (v + N, u)$$

for $v < u < t < v + N$.

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