ON FINITE SOLVABLE LINEAR GROUPS

BY

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Introduction

The following theorem of N. Itô [3] is well known.

Let G be a finite solvable group and let p be a prime. Let G have a faithful representation of degree n over the complex number field. If $n < p$, then G has a normal abelian p-Sylow subgroup unless p is odd, $n = p - 1$ and p is a Fermat prime.

We shall prove here the following generalization.

Theorem. *Let G be a finite solvable group, p a prime. Let G have a faithful irreducible representation of degree n over the complex number field. Then G has an abelian normal p-Sylow subgroup unless, for some positive integer m, $n = mp$ or $n = mq^s$ where q^s is a positive power of a prime with $q^s \equiv \pm 1 \pmod p$.*

The theorem is "best possible" in the sense that for each of the exceptional values of n mentioned there is a finite solvable group which has a faithful irreducible representation of degree n over the field of complex numbers and which does not have a normal p-Sylow subgroup (§2). The proof of the theorem is not applicable to p-solvable groups as is the proof given by Itô for his theorem and leaves open the question as to whether or not there is an analogue for p-solvable groups. The reader is referred to [1] for another kind of generalization of Itô's theorem. We shall make frequent use of the following statement which is a well known consequence of Clifford's theorem and the Frobenius reciprocity theorem.

Lemma. *Let H be a normal subgroup of prime index p of the finite group G and let θ be an irreducible complex character of H with inertia group $I(\theta)$ in G. If $I(\theta) = H$, then the induced character θ^* is an irreducible character of G and is the only irreducible character of G whose restriction to H contains θ; $\theta^* \mid H$ is a sum of p distinct conjugate characters of H. If $I(\theta) = G$, then there are p distinct characters of G which are extensions of θ and these are the only irreducible characters of G whose restriction to H contains θ.*

1. Proof of the theorem

Let G be a counterexample to the theorem of minimal order. A contradiction is obtained after a series of steps. Let χ denote the given faithful irreducible character of G.

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