THE ADJOINT FUNCTOR THEOREM AND THE YONEDA EMBEDDING

BY

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The aim of this note is to show that the problem of whether direct limit preserving functors $T : \mathfrak{A} \to \mathfrak{A}'$ ($\mathfrak{A}$ fixed) have right adjoints is equivalent to the problem of whether the inverse limit preserving Yoneda embedding $Y : \mathfrak{A} \to \text{Cont}\,[\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$, $A \rightsquigarrow [-, A]$, has a left adjoint, where Cont $[\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$ denotes the category of contravariant set valued functors which take direct limits in inverse limits (also called continuous functors). In other words, the problem of constructing right adjoints of functors with domain $\mathfrak{A}$ can be transformed into the problem of constructing the left adjoint of a functor with domain $\mathfrak{A}$. If the conjugate t^* of every covariant and contravariant continuous functor $t : \mathfrak{A} \to \mathfrak{S}$ is again a functor[2], then it follows from this that co-continuous functors $T : \mathfrak{A} \to \mathfrak{A}'$ have right adjoints iff continuous functors $S : \mathfrak{A} \to \mathfrak{A}''$ have left adjoints ($\mathfrak{A}$ fixed, $\mathfrak{A}'$ and $\mathfrak{A}''$ variable). This gives rise to a simple proof of the adjoint functor theorem which, like the proofs of J. Benabou, J. Beck, P. Dedecker, J. Isbell, J. Lambek and others, does not require that $\mathfrak{A}$ is well or co-wellpowered in the sense of P. Freyd [1]. In other words, the subobjects and the quotient objects of an object in $\mathfrak{A}$ need not form a set.

We were led to the above after observing a new proof of Lambek's version [4] of the special adjoint functor theorem. According to Lambek, Freyd's conditions in [1] that $\mathfrak{A}$ has a family of generators and is co-wellpowered can be replaced by requiring that $\mathfrak{A}$ has a small adequate or dense subcategory $\bar{\mathfrak{A}}$ (cf. Isbell [3], Ulmer [5, 1.3]). Adequate or dense means that every object $A \in \mathfrak{A}$ is the direct limit of the canonical diagram of objects $\bar{A} \in \bar{\mathfrak{A}}$ over A. (More precisely, the objects of the index category are morphisms $\iota : \bar{A}_\iota \to A$ in $\mathfrak{A}$, where $\bar{A}_\iota \in \bar{\mathfrak{A}}$. A morphism $\alpha : \iota \to \kappa$ is a morphism $\alpha : \bar{A}_\iota \to \bar{A}_\kappa$ in $\bar{\mathfrak{A}}$ with the property $\iota = \kappa \cdot \alpha$.)

Morphism sets, natural transformations and functor categories are denoted by brackets $[-, -]$, comma categories by parentheses $(-, -)$. The category of sets is denoted by $\mathfrak{S}$. The phrase "Let $\mathfrak{A}$ be a category with direct limits" always means that $\mathfrak{A}$ has direct limits over small index categories. However, we sometimes also consider direct limits of functors $F : \mathfrak{D} \to \mathfrak{A}$, where $\mathfrak{D}$ is not necessarily small. Of course we then have to prove that this specific

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[2] Recall that $t^*A = [t, [A, -]]$ for $A \in \mathfrak{A}$. Thus t^* is a functor iff the natural transformations from t to $[A, -]$ form a set for every $A \in \mathfrak{A}$ (likewise for contravariant functors).