ON CERTAIN ELEMENTS OF C^*-ALGEBRAS

BY

J. A. ERDOS[1]

1. Introduction

Let A be a C^*-algebra; that is, a Banach algebra with involution satisfying the C^*-condition $\| aa^* \| = \| a \|^2$. It is well known that A can be faithfully represented as an algebra of operators on some Hilbert space. However it is clear that the rank of the image of a given element may vary in different representations. The main purpose of this paper is to give a necessary and sufficient condition for an element to have an operator of rank one as its image under some faithful representation of A.

We shall call an element s of a C^*-algebra A a *single* element if, whenever $asb = 0$ for some a, b in A, we have that at least one of as, sb is zero.

It is easy to see that an operator of rank one is a single element of any algebra of operators that contains it. The condition of being single has been used by Ringrose [8], [9] as a property of rank one operators in triangular and nest algebras that is invariant under algebraic isomorphisms. Munn [6] has also used the condition, imposing it on every element of an algebraic semigroup, but there does not seem to be any obvious connection between [6] and what follows here.

It will be shown that there exists a faithful representation of A such that the image of every non-zero single element of A is an operator of rank one. This is done in Sections 2 and 3. In Section 4 the theory developed is applied to give the standard representation of a dual C^*-algebra. This section also contains characterizations of dual C^*-algebras and certain W^*-algebras, including a characterisation of the algebra of all bounded linear operators on a Hilbert space. The final section contains an example showing that the main result cannot be generalised to all Banach algebras.

In general, the terminology used will be as in Rickart [7] and Dixmier [1], [2]. At certain points, noted in the text, representations that are not adjoint preserving will be considered. Otherwise representations will be, by definition adjoint preserving. All the algebras considered will be over the complex field.

2. Algebraic properties

The set of single elements of the C^*-algebra A will be denoted by σ. Note that the zero element is a member of σ.

LEMMA 2.1. *If* $s \in \sigma$ *then* $s^* \in \sigma$. *For any element* x *of* A, $xs \in \sigma$ *and* $sx \in \sigma$.

Received May 2, 1969.

[1] The results of this paper were obtained while the author was a Research Associate at the University of Illinois.