FIXED-POINT THEOREMS FOR CERTAIN NONLINEAR NONEXPANSIVE MAPPINGS

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1. Introduction

Let X be a Banach space, K a bounded closed and convex subset of X, and T a *nonexpansive mapping* of K into itself (thus $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in K$). In this paper we treat the problem of determining what conditions on T will assure existence of fixed point for T when K is weakly compact.

Rather extensive study of fixed-point theory for nonlinear nonexpansive mappings was initiated with the proof that if K is a bounded closed convex subset of a uniformly convex Banach space X (or more generally, if X is reflexive and K possesses "normal structure"), then a nonexpansive mapping T: $K \rightarrow K$ always has a fixed point (see Browder [3], Göhde [6], Kirk [7]). In order to remove the geometric assumptions (either uniform convexity of X or normal structure of K) so that fixed point theorems for nonexpansive mappings might be obtained which hold in *all* reflexive spaces, additional assumptions must be made about the mapping T. This motivated the introduction of the concept of "diminishing orbital diameters" in Belluce-Kirk [1].

Let $A \subset X$ and F a mapping of A into itself. For $x \in A$, let

$$O(F^n x) = \{F^n x, F^{n+1} x, F^{n+2} x, \cdots\}, \quad n = 0, 1, 2, \cdots$$

(where $F^0 x = x$), and for $B \subset A$, let

$$\delta B = \sup \{ \| x - y \| : x, y \in B\}$$

denote the *diameter* of B. A point $z \in B$ is called a *nondiametral* point of B if $\sup \{ \| z - y \| : y \in B\} < \delta B$.

DEFINITION [1]. The mapping F: $A \rightarrow A$ is said to have *diminishing orbital diameter* (d.o.d.) at $x \in A$ if either $\delta O(x) = 0$ or

$$\lim_{n \to \infty} \delta O(F^n x) < \delta O(x).$$

F has *diminishing orbital diameters* on A if F has d.o.d. at each point of A.

Nonexpansive mappings which have diminishing orbital diameters include mappings f: $A \rightarrow A$ which satisfy:

(I) *For each $x \in A$ there is a number $\alpha(x) < 1$ such that*

$$\| Fx - Fy \| \leq \alpha(x) \| x - y \|$$

for each $y \in A$.

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