THE NORMAL INDEX OF MAXIMAL SUBGROUPS IN FINITE GROUPS

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In [4] Deskins defined the normal index of a maximal subgroup M in a finite group G as the order of a chief factor H/K of G where H is minimal in the set of normal supplements to M in G. We let $\eta(G:M)$ denote this number. The following two results relating to normal index were announced by Deskins [4].

(A) The finite group G is solvable if and only if for each maximal subgroup M of G, $\eta(G:M)$ is a power of a prime.

(B) The finite group G is solvable if and only if $\eta(G:M) = [G:M]$ for each maximal subgroup M of G.

In this note we obtain (B) as a corollary to a theorem on p-solvability. We also show that if G has at least one solvable maximal subgroup M such that $\eta(G:M) = [G:M]$, then G is solvable. The authors would like to thank Professor Deskins for some comments helpful in the preparation of this paper. All groups are assumed to be finite.

We begin with a lemma stated by Deskins [4, 2.1] and proved here for the sake of completeness.

LEMMA 1. *$\eta(G:M)$ is uniquely determined by M.*

Proof. We wish to show that if H_1 and H_2 are minimal in the set of normal supplements to M in G and K_1 and K_2 are maximal G-subgroups of H_1 and H_2 respectively, then $|H_1/K_1| = |H_2/K_2|$. The proof is by induction on $|G|$. By the minimality of H_i, $K_i \leq M$, $i = 1, 2$, so if $K_1 \cap K_2 \neq \langle 1 \rangle$, the result follows by induction. Thus we may suppose that $K_1 \cap K_2 = \langle 1 \rangle$. We note that

$$H_1 \cap K_2 \lhd G \quad \text{and} \quad H_1 \cap K_2 \leq H_1 \cap M$$

so $H_1 \cap K_2 \leq K_1$. Thus $H_1 \cap K_2 \leq K_1 \cap K_2 = \langle 1 \rangle$. Similarly, $H_2 \cap K_1 = \langle 1 \rangle$. In $G/K_1 K_2$, $H_1 K_2/K_1 K_2$ is minimal in the set of normal supplements to $M/K_1 K_2$. Certainly $H_1 K_2/K_1 K_2$ is a supplement, so suppose $X/K_1 K_2$ is a normal supplement to $M/K_1 K_2$ with $H_1 K_2/K_1 K_2 \geq X/K_1 K_2$. Then

$$(X \cap H_1)M = (X \cap H_1)K_2 M = (XK_2 \cap H_1 K_2)M$$
$$= (X \cap H_1 K_2)M = XM = G.$$

Received July 7, 1969.

[1] This author was supported by a National Science Foundation grant.