MOMENTS OF RANDOM WALK HAVING INFINITE VARIANCE AND THE EXISTENCE OF CERTAIN OPTIMAL STOPPING RULES FOR S_n/n

BY

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Let $X_1, X_2, \cdots$ be independent identically distributed random variables with mean 0, $S_n = X_1 + \cdots + X_n$. If $\int |X_1|^q < \infty$ for some $q \geq 2$ the asymptotic behavior of the distributions of S_n is known to be very regular. One crude indicant of this is the fact that $\sup_n \| S_n \|_q / \| S_n \|_1 < \infty$. This is proved in [1]. If $\int |X_1|^q < \infty$ for some q between 1 and 2 but $\int |X_1|^2 = \infty$, the situation considered here, the behavior in this norm sense of the distributions of S_n can be much worse. An example will be given to show that, for any q between 1 and 2, $\int |X_1|^q$ can be finite but $\lim \| S_n \|_q / \| S_n \|_1 = \infty$. However, for any such example it is proved that $\int |X_1|^{q+\varepsilon} = \infty$, $\varepsilon > 0$. That is, if $1 < \alpha < \beta < 2$ and $\int |X_1|^\beta < \infty$ then $\liminf \| S_n \|_\alpha / \| S_n \|_1 < \infty$. The lim sup need not be finite and an example is given to show this.

Using this small amount of regularity which does exist it is then proved that if some absolute moment of X_1 higher than the first is finite then an optimal stopping rule exists for S_n/n, verifying a conjecture made by Dvoretzky in [4]. The existence of such a rule when $\operatorname{Var} X_1 < \infty$ has been shown by Dvoretzky in [4] and by Teicher and Wolfowitz in [9]. Some results if $\operatorname{Var} X_1 = \infty$ appear in [11]. It was very helpful to see a copy of Mary Thompson's thesis, [10], before its publication.

1. Moments of S_n

In what follows $p < q$ will be numbers between 1 and 2, X will be a random variable satisfying $E|X|^q < \infty$, $EX = 0$, $\operatorname{Var} X = \infty$, and $X_1, X_2, \cdots$ will be independent random variables each having the distribution of X. $C_1, C_2, \cdots$ will be positive constants depending only on p and q.

The idea of the following lemma is well known.

LEMMA 1. *Let f be a nonnegative random variable and $1 \leq a < b < \infty$. Then*

$$P(f > \|f\|_a/2) \geq 2^{b/(a-b)} (\|f\|_a / \|f\|_b)^{ab/(b-a)}.$$

Also,

$$P(f > \|f\|_a/2) \geq \|f\|_a^a / 2 \|f\|_\infty^a .$$

Proof. Let $E = \{f > \|f\|_a/2\}$. Using Holder's inequality,

$$\|f\|_a^a/2 \leq \int_E f^a \leq \|f^a\|_{b/a} \| I(E) \|_{b/(b-a)} = \|f\|_b^a P(E)^{(b-a)/b},$$

Received September 14, 1970.

[1] Research partially supported by a National Science Foundation grant.