A NON-CRITERION FOR CENTRAL SIMPLE ALGEBRAS

BY

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Let k be a field, K an extension field, and U a finite-dimensional k-algebra containing K. Sweedler [2] proved that if U is central simple with K as maximal commutative subfield, then U is isomorphic to $K \otimes K$ as a $K - K$ bimodule; for K/k purely inseparable, he also proved the converse. Chase [1] has given simpler proofs of these results, and has also shown that the converse fails with K/k normal separable. In this note I show that the converse fails whenever K/k is not purely inseparable.

First recall some of the results of [2]. Let ε_j $(j = 1, 2, 3, 4)$ be the natural algebra maps of $K \otimes K \otimes K$ into $K \otimes K \otimes K \otimes K$ given by inserting a 1 in the j^{th} place. (For brevity we write $\otimes$ instead of $\otimes_k$.) An element $x = \sum a_i \otimes b_i \otimes c_i$ is called a *cocycle* if

(1) $\sum a_i b_i \otimes c_i = e \otimes 1$ and $\sum a_i \otimes b_i c_i = 1 \otimes e$ for some $e \neq 0$, and
(2) $\varepsilon_1(x)\varepsilon_3(x) = \varepsilon_2(x)\varepsilon_4(x)$.

Let U be the algebra $\text{End}_k K$, with K embedded in it as the multiplication operators. Given a cocycle x, define a new multiplication on U by $u * v = \sum a_i u b_i v c_i$. Then this gives an associative k-algebra containing K and isomorphic to $K \otimes K$ as a $K - K$ bimodule, and every such algebra arises in this way. The algebra is central simple iff x is invertible (and is therefore an Amitsur cocycle), and every central simple algebra with K as maximal commutative subfield arises in this way.

Our problem thus is to show that not all cocycles are invertible. For K/k separable we will in fact write down a cocycle x which is a nontrivial idempotent. If L is a superfield of K, the image of x in $L \otimes L \otimes L$ will again be a cocycle and a nontrivial idempotent. Since any extension not purely inseparable contains a separable subextension, this will complete the proof.

We therefore assume K/k separable. The kernel of the multiplication map $K \otimes K \to K$ is then generated by an idempotent f, and we set

$$x = 1 \otimes 1 \otimes 1 - (f \otimes 1)(1 \otimes f).$$

It is easy to verify that this is a nontrivial idempotent satisfying condition (1) with $e = 1$.

To check condition (2) we compute in the Galois closure E of K. If $\sigma_1, \cdots, \sigma_4$ run independently over the maps of K into E, then

$$a \otimes b \otimes c \otimes d \mapsto \sigma_1(a)\sigma_2(b)\sigma_3(c)\sigma_4(d)$$

runs over the maps of $\otimes^4 K$ into E, and the idempotents $\varepsilon_j(x)$ always map to 0

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