IMAGES OF BILINEAR SYMMETRIC AND SKEW-SYMMETRIC FUNCTIONS

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1. Introduction

Let $U$, $V$ and $W$ be vector spaces over a field $F$ and let $\varphi : U \times V \rightarrow W$ be a bilinear function. We define the image of $\varphi$ to be the set of all vectors in $W$ of the form $\varphi(x, y)$, $x \in U$, $y \in V$ and denote it by $\text{Im} \, \varphi$. It is not generally the case that $\text{Im} \, \varphi$ is a subspace of $W$. In the paper [2] the following result is proved by the first author.

THEOREM 1. Let $V_1$ and $V_2$ be vector spaces of dimensions $n_1$ and $n_2$ respectively, $n_1 \leq n_2$. If $\varphi$ is a bilinear function on $V_1 \times V_2$ such that $\text{Im} \, \varphi$ is a vector space then

$$\dim (\text{Im} \, \varphi) \leq n_1(n_2 - 1) - \left[\frac{1}{2} - \sqrt{(n_1 + 5/4)}\right]$$

where $[x]$ denotes the greatest integer function.

In this paper we consider this problem for bilinear symmetric and skew-symmetric functions. The main results follow.

THEOREM 2. Let $F$ be an algebraically closed field of characteristic 0 and let $V$ be an $n$-dimensional vector space over $F$. If $\varphi$ is a bilinear symmetric function defined on $V \times V$ such that $\text{Im} \, \varphi$ is a vector space $U$ then

$$(1) \quad \dim (U) \leq n(n + 1)/2 - \left[\frac{1}{2}(n + 1 - \sqrt{(n + 3)})\right].$$

THEOREM 3. Let $\varphi$ be a bilinear skew-symmetric function defined on $V \times V$, where $V$ is an $n$-dimensional vector space over a field $F$ of characteristic 0. If $\text{Im} \, \varphi$ is a vector space then

(i) $\text{Im} \, \varphi = \{0\}$ if $n = 1$, and
(ii) $\dim (\text{Im} \, \varphi) \leq n(n - 1)/2 - \left[\frac{1}{2}(n - \sqrt{(n + 2)})\right]$ if $n \geq 2$.

Some examples follow that show that if $\varphi$ is a bilinear, symmetric or skew-symmetric function then the image of $\varphi$ may or may not be a vector space.

**Example 1.** Let $U$ and $V$ be vector spaces over a field $F$ and let $T : V \rightarrow U$ be a linear transformation. Let $f \in V^*$ be a non-zero linear functional. Define $\varphi : V \times V \rightarrow U$ by

$$\varphi(x, y) = f(x)Ty + f(y)Tx, \quad x, y \in V.$$