THE EQUICONTINUOUS STRUCTURE RELATION FOR ERGODIC ABELIAN TRANSFORMATION GROUPS

BY

KENT E. WESTERBECK AND TA-SUN WU[1]

I. Introduction

Let $(\tilde{X}, T)$ be a transformation group with compact, metric phase space, $\tilde{X}$, and abelian phase group, T. $(\tilde{X}, T)$ is ergodic if every proper, closed, T-invariant subset is nowhere dense. By [7] this is equivalent to requiring the set of points, X, whose orbits are dense in $\tilde{X}$, to be comeager. $(\tilde{X}, T)$ is weakly mixing if $(\tilde{X} \times \tilde{X}, T)$ is ergodic, where the action of T is given by $(x, x')t = (xt, x't)$.

In [2], it was shown that there exists on $(\tilde{X}, T)$, a least, closed, T-invariant equivalence relation, $\tilde{S}_e$, such that $(\tilde{X}/\tilde{S}_e, T)$ is an equicontinuous transformation group. $\tilde{S}_e$ is called the equicontinuous structure relation on $\tilde{X}$. In [15], Veech made a thorough study of $\tilde{S}_e$ when $(\tilde{X}, T)$ is a minimal set. However, when $(\tilde{X}, T)$ is not minimal, the relation $\tilde{S}_e$ could be quite obscure. Consider, for example, the continuous flow acting on the unit interval with two end points fixed. Then $\tilde{S}_e = \tilde{X} \times \tilde{X}$. If we restrict our attention to the subflow (X, T), where X is the open interval, then there is a faithful homomorphism of (X, T) into the universal almost periodic minimal set. On the other hand, consider the Stepanoff flows on the two torus with one fixed point [13]. In this case, $\tilde{S}_e$ is again equal to $\tilde{X} \times \tilde{X}$, but in some instances, (X, T) cannot be mapped homomorphically into any nontrivial almost periodic minimal flows. The differences between these two examples seem to indicate it is more natural to consider the homomorphisms from (X, T) into almost periodic minimal flows with compact phase space, when $(\tilde{X}, T)$ is ergodic and nonminimal. In this note, we shall prove the existence of a least, closed, invariant equivalence relation, S_e, on (X, T) such that there exists a faithful homomorphism of $(X/S_e, T)$ into a compact, almost periodic, minimal transformation group with a certain universality property. We will demonstrate a condition on $(\tilde{X}, T)$ equivalent to the existence of an invariant, Borel, probability measure on $(\tilde{X}, T)$ with support $\tilde{X}$. Assuming one of these conditions, we will characterize S_e, and show it is contained in the regional proximal relation on (X, T) [2]. Finally, as applications, we will show the eigenfunctions and spatial eigenfunctions of Keynes and Robertson [11] are essentially equal and will give a sufficient condition for $(\tilde{X}, T)$ to be weakly mixing.

II. Construction of an almost periodic, minimal factor of (X, T)

Standing Notation. Throughout this paper $(\tilde{X}, T)$ will denote an ergodic transformation group with compact, metric phase space, $\tilde{X}$, and abelian phase

Received July 20, 1971.

[1] The second author's work was supported by a National Science Foundation grant.