A COMBINATORIAL IDENTITY WITH APPLICATIONS TO REPRESENTATION THEORY

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Introduction

We apply a combinatorial identity (Proposition 1) to obtain two results. First we characterize the class of finite groups such that similar permutation representations contain the regular representation the same number of times (Proposition 2), thus answering a question of S. Golomb [9]. Secondly we obtain a relatively short and direct proof of the Brauer-Tate and Witt-Berman theorems (Proposition 3).

The proof of these results uses the Möbius function of the lattice of subgroups of a group. P. Hall [4] and L. Weisner [8] have extended the Möbius inversion formula from functions of integers to functions defined on locally finite partially ordered sets, and their work has been generalized by G.-C. Rota [7]. (A partially ordered set is locally finite if each interval $[a, b] = \{c \mid a \leq c \leq b\}$ is finite.) The original Möbius function is multiplicative on relatively prime numbers, and Proposition 1 is an extension of multiplicativity to a relation for the Möbius function of locally finite lattices.

Proposition 1 has also been obtained by Curtis Greene [10].

A combinatorial identity

The Möbius function μ of a locally finite partially ordered set is the unique solution of

$$\begin{aligned} \textstyle\sum_{c \in [a,b]} \mu(c, b) &= 1 \quad \text{if } a = b \\ &= 0 \quad \text{if } a \neq b. \end{aligned} \tag{1}$$

See [7] for a complete exposition. By [7, §3 Proposition 3], μ also satisfies

$$\begin{aligned} \textstyle\sum_{c \in [a,b]} \mu(a, c) &= 1 \quad \text{if } a = b \\ &= 0 \quad \text{if } a \neq b. \end{aligned} \tag{1$'$}$$

Proposition 1. *Let L be a locally finite lattice with Möbius function μ. Suppose $c \in [a, b]$. Then*

$$\mu(a, b) = \textstyle\sum_{d \vee c = b, d \wedge c = a} \tilde{\mu}(a, d)\mu(d, b) \tag{2}$$

where $\tilde{\mu}$ is the Möbius function of

$$[a, d]^* = \{e \mid e \in [a, d], (e \vee c) \wedge d = e\}.$$

An empty sum is zero.

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