ISOMETRIES OF FUNCTION ALGEBRAS

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Let X and Y be compact Hausdorff spaces. A and B will denote sub-algebras of $C(X)$ and $C(Y)$ respectively. ($C(X)$ indicates the space of continuous complex-valued functions on X.) It will be assumed that A and B are equipped with the sup-norm, are point separating, and contain the constant functions. In this paper, we give a description of the linear isometries from A to B in the case where $A = C(X)$ and $B = C(Y)$, and under certain restrictions on the pair (X, Y).

Operators of the form

$$(*) \qquad Tf = g(f \circ \psi),$$

where g is a fixed function in $C(Y)$ of norm 1 and ψ is a continuous map from Y into X such that $\psi(|g|^{-1}(1)) = X$, constitute a class of isometries from $C(X)$ into $C(Y)$. In fact, if T is an isometry of $C(X)$ onto $C(Y)$, then T must be of the form $(*)$ (see, e.g., [1, p. 442]). It is not true, in general, that all isometries from $C(X)$ into $C(Y)$ are of the form $(*)$. For example: let $\phi_i : [0, 1] \to [0, 1]$, $i = 1, 2$ be continuous functions having the following properties: $\phi_1 = \phi_2$ on $[0, 1/2]$, $\phi_1([0, 1/2]) = [0, 1]$, and $\phi_1(1) \neq \phi_2(1)$. Define isometries $T_i : C[0, 1] \to C[0, 1]$ by $T_i f = f \circ \phi_i$, $i = 1, 2$. Let

$$T = (1/2)T_1 + (1/2)T_2 .$$

Then T is an isometry, but T is not of the form $(*)$.

Let S_A and S_B denote the unit balls in the dual spaces of A and B respectively. Suppose $T : A \to B$ is an isometry. It follows from the Hahn-Banach theorem, that the adjoint T^* of T maps S_B onto S_A. Let l be an element of the set ex S_A of extreme points of S_A. Then $(T^*)^{-1}(l) \cap S_B$ is a non-empty weak* closed face of S_B. (A *face* F of a convex set K is a convex subset of K such that

$$cf_1 + (1 - c)f_2 \in F \quad \text{and} \quad (c, f_1, f_2) \in (0, 1) \times K \times K$$

implies that $f_1, f_2 \in F$.) It follows from the Krein-Milman Theorem that there is an extreme point e of S_B such that $T^*(e) = l$. It is known (see, e.g., [3, Prop. 6.2]) that l is an extreme point of S_A iff it is of the form $e^{i\alpha}l_x$, where $\alpha \in [0, 2\pi]$ and l_x denotes evaluation at a point x of the Choquet boundary of X with respect to A. Thus, we have the following:

PROPOSITION 1. *Let T be an isometry from A into B. Let $Y(T) = \{y \in Y \mid |T1(y)| = 1$ and there is a $\hat{T}(y) \in X$ such that $Tf(y) =$*

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