A NECESSARY AND SUFFICIENT CONDITION FOR THE RIEMANN HYPOTHESIS FOR RAMANUJAN'S ZETA FUNCTION

BY

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1. Introduction

The Ramanujan τ-function is defined as the *n-th* coefficient in the q-expansion of

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \qquad q = e^{2\pi i z} \text{ and } \operatorname{Im}(z) > 0.$$

It is well known that the function $\Delta(z)$ spans the space of cusp forms of weight -12 associated with the unimodular group. $\Delta(z)$ is in fact an eigenfunction of the Hecke operators and as such its corresponding Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \tau(n) n^{-s}$$

has an Euler product

$$\varphi(s) = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}.$$

In [4, p. 174], Hardy observed that the location of the nontrivial zeros of $\varphi(s)$ in the strip $11/2 \leq \operatorname{Re}(s) \leq 13/2$ gave rise to problems similar to those in the classical theory of the Riemann zeta function, i.e., Riemann Hypothesis, von Mangoldt formulae etc. Some of these questions were subsequently treated by various authors. Of particular interest is the paper by Goldstein [2] where the analogue of Merten's conjecture

$$\sum_{n \leq x} \mu(n) \ll x^{1/2+\varepsilon}$$

is established for Ramanujan's zeta function. A special case of Goldstein's main result is the

THEOREM (3.6 in [2]). *A necessary and sufficient condition for Ramanujan's zeta function $\varphi(s)$ to have all its zeros on* $\operatorname{Re}(s) = 6$ *is that*

$$(1) \qquad \sum_{n \leq x} \mu_\varphi(n) \ll x^{6+\epsilon}$$

for all $\varepsilon > 0$, where $\mu_\varphi(n)$ is the Möbius function defined by expanding formally the product

$$(2) \qquad {}_x\prod_p (1 - \tau(p)p^{-s} + p^{11-2s}) = \sum_{n=1}^{\infty} \tau(n) n^{-s}.$$

In [2] Goldstein also suggested that the arithmetical function $\mu_\varphi(n)$ could be evaluated by means of the Selberg trace formula [7] in terms of ideal class numbers of certain imaginary quadratic fields and thus transform the sum in

Received March 17, 1972.