ON THE DENSITY OF CONJUGATE FIXED POINTS OF A CORRESPONDENCE

BY

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1. Introduction

In connection with another investigation the following question arose. Suppose that $P(x, z)$ is an irreducible polynomial in two variables over the finite field $k = GF(q)$. Let the pair (α, β) lie in the graph of P, that is let $P(\alpha, \beta) = 0$ where α and β are elements of the algebraic closure $\bar{k}$ of k. We now ask: how often does it occur that α and β are conjugate over k, that is, how often is $\beta = \alpha^{q^r}$ for some r? For example, how often does $\alpha + 1 = \alpha^{q^r}$? It is immediately clear that this situation will arise infinitely often for a given P so the question of "how often" must be answered in terms of a density. To attack this question we will find it useful to consider $P(x, z)$ as a correspondence from the projective line to itself. Then to say that $P(\alpha, \beta) = 0$ is to say (with finitely many exceptions) that there are points $\mathcal{P}_1$ and $\mathcal{P}_2$ on the projective line over k such that $x(\mathcal{P}_1) = \alpha$ and $z(\mathcal{P}_2) = \beta$. To say that α and β are conjugate is equivalent to saying that $\mathcal{P}_1 = \mathcal{P}_2$. More generally, let us replace the projective line by the Riemann surface $X^{K/k}$ for any function field K in one variable in which $k = GF(q)$ is the exact constant field. When P is any prime correspondence from K to K and $\mathcal{P}$ is in $X^{K/k}$ then we may let P act on $\mathcal{P}$ and produce an integral divisor $\mathcal{P}^P$ also on K. We are, therefore, interested in the set

$$A_p = \{\mathcal{P} \in X^{K/k} \mid \mathcal{P} \text{ divides } \mathcal{P}^P\}.$$

In order to measure the size of A_P in $X^{K/k}$ we use the ordinary Dirichlet density $\delta(A_p)$. It is the purpose of this paper to prove the following result.

Theorem. *If* (i) *both degrees of P are one and P is not the diagonal or* (ii) *the ratio of the degrees of P is not an integral power of q, then $\delta(A_p) = 0$.*

It seems quite likely that the above theorem is true more generally when P is assumed to be unequal to any power of the Frobenius correspondence or its Rosati adjoint. The methods available to us unfortunately break down when both degrees of P are equal but unequal unity.

The paper will consist of three more sections. The first of these deals with generalities concerned with correspondences and the Dirichlet density. Our treatment of the former is completely algebraic since the basic rationality questions involved make such a treatment more useful than the customary

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