A NON-NORMAL HEREDITARILY-SEPARABLE SPACE

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Let us for the purposes of this paper use *S-space* to mean a hereditarily-separable regular Hausdorff space.

If an S-space is not normal, it is clearly not Lindelöf. Although both unfortunately depend on special set-theoretic assumptions, recently [1], [2] examples have been given of non-Lindelöf S-spaces; both happen to be normal.

So there is current vogue for the question, which Jones [3] says is an old one: *Is every S-space normal?* We prove here that the answer is at least conditionally *no*. Jones [3] shows a non-normal S-space can be used to construct a non-completely regular S-space. Thus it is consistent with the usual axioms of set theory that there be a non-completely regular S-space.

Let us call Σ an S^*-space provided Σ is an uncountable S-space with a basis for its topology consisting of sets which are open, closed, and countable. Clearly no S^*-space is Lindelöf.

The space described in [1] is an S^*-space and this space exists if there is a Souslin line.

In recent correspondence I. Juhász and J. Gerlits point out the following.

THEOREM 1. *If there is a Souslin line* (which is consistent with the axioms of set theory), *then there is a non-normal S-space.*

Proof. Let Σ be the S-space described in [1]. Let I be the closed unit interval. Using the precise technique given in [5] construct from Σ a normal space T such that $T \times I$ is not normal. Then $T \times I$ will be a non-normal S-space.

A perhaps more general construction gives the following.

THEOREM 2. *Assume that there is an S^*-space and $2^{\aleph_0} < 2^{\aleph_1}$.* (This combination is consistent with the usual axioms of set theory, being true, for instance, in $V = L$, Gödel's constructible model of the universe.) *Then there is a non-normal S-space.*

Proof. We use the following pretty lemma of F. B. Jones [4]. This whole paper is an excuse to restate this lemma.

LEMMA. *There exists a cardinality $\aleph_1$ subset A of the real numbers such that each countable subset B of A is a relative G_δ set. Observe that B countable implies $A - B$ is a G_δ but $2^{\aleph_0} < 2^{\aleph_1}$ implies there is a subset C of A which is not a G_δ.*

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