UNIFORM SPLINE INTERPOLATION OPERATORS IN L_2

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1. Let $m \geq 1$ and $n \geq 2$ be positive integers. Define the class $\mathcal{S}^{2m}$ of cardinal splines of degree $2m - 1$ to be those functions S satisfying

(i) S is a polynomial of degree at most $2m - 1$ on each of the intervals

$$[i, i + 1], i = 0, \pm 1, \pm 2, \cdots \tag{1}$$

(ii) $S \in C^{2m-2}(-\infty, \infty)$

If in addition

$$S(x + n) = S(x) \quad \text{all} \quad x, \tag{2}$$

we say S is a periodic spline and denote this class by $\mathcal{S}_n^{2m}$. Let $l_p(n)$ be the space of real n-tuples possessing the norm

$$\| y \|_p = (\textstyle\sum_{i=1}^{n} | y_i |^p)^{1/p}, \qquad 1 \leq p < \infty$$
$$= \max_{1 \leq i \leq n} | y_i |, \qquad p = \infty.$$

Then the periodic spline interpolation operator $\mathcal{L}_n^{2m} : l_p(n) \to L_p[0, n]$ is defined by letting $\mathcal{L}_n^{2m} y$ be that unique element of $\mathcal{S}_n^{2m}$ satisfying

$$\mathcal{L}_n^{2m} y(i) = y_i, \qquad i = 1, 2, \cdots, n.$$

Similarly if $(y_i)_{i=-\infty}^{\infty}$ is in l_p, the class of doubly infinite real p — summable sequences, then the cardinal spline interpolation operator

$$\mathcal{L}^{2m} : l_p \to L_p(-\infty, \infty)$$

is defined by letting $\mathcal{L}^{2m} y$ be that unique element of $\mathcal{S}^{2m} \cap L_p(-\infty, \infty)$ which satisfies

$$\mathcal{L}^{2m} y(i) = y_i, \qquad i = 0, \pm 1, \pm 2, \cdots.$$

The problem of calculating the norms of these operators for $p = \infty$ was first posed by Schurer and Cheney [6], who obtained

THEOREM 1 (Schurer and Cheney). *Let* $\beta = 2 + \sqrt{3}$. *Then*

$$\| \mathcal{L}_n^4 \|_\infty = 1 + \tfrac{3}{2}(\beta^k - \beta)(\beta^k + 1)^{-1}(\beta - 1)^{-1}, \qquad n = 2k$$
$$= 1 + \tfrac{3}{2}(\beta^k - \beta)(\beta^k + \beta)(\beta^{2k} + \beta)^{-1}(\beta - 1)^{-1}, \; n = 2k - 1, \tag{3}$$
$$\| \mathcal{L}^4 \|_\infty = (1 + 3\sqrt{3})/4.$$

Solutions were later obtained by Schurer [5] for $m = 3$ and Richards [1] for

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