IMMERSIONS UP TO COBORDISM

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Given a compact manifold M^m we ask for the least integer k such that M^n immerses into R^{m+k}. A great deal is known for special classes of manifolds (see Gitler [9]). There is a conjecture that (if $m \geq 2$) M^m immerses in $R^{2m-\alpha(m)}$, where $\alpha(m)$ is the number of ones in the dyadic expansion of m. The original question can be weakened to read: given M^m, find the least integer k such that there is a manifold M' cobordant to M and M' immerses in R^{m+k}.

We shall say that M^m immerses into R^{m+k} up to cobordism. Brown [4], [5] has proved that M^m immerses in $R^{2m-\alpha(m)}$ up to cobordism. Of course, we have lost a lot of geometric information by passing to cobordism, since now $k(M)$ is a function only of the cobordism class of M, and if M is a boundary ($M = RP^{2n+1}$, an odd-dimensional real projective space, for example) then $k(M) = 0$. Even if M is not a boundary, a manifold may immerse up to cobordism into a lower dimensional Euclidean space than M itself: for example, RP^{10} immerses up to cobordism into R^{15}, as we shall see later, but RP^{10} itself immerses into R^{16}, and does not immerse into R^{15} (see Gitler [9]).

The fact that we have lost geometric information by passing to cobordism (and reducing the problem to homotopy theory) should not make us sad: the geometric situation was too complicated, so we would not obtain useful qualitative information if we preserved the complexity of the original problem. The purpose of this note is to convince the reader that even after the reduction there is a lot of structure (possibly even too much?) remaining.

The usual approximation theorems of Thom [16] give a reduction of the problem of immersions up to cobordism to a question of homotopy. Let MO be the Thom spectrum [16] for the orthogonal group, then cobordism classes of compact m-dimensional manifolds correspond to elements of

$$\pi_m(MO) = \varinjlim_n \pi_{n+m}(MO(n)).$$

Let $\lambda_k\colon \pi^{st}_{m+k}(MO(k)) \to \pi_m(MO)$ be the map into the direct limit, where the superscript st denotes stable homotopy. If $x \in \pi_m(MO)$ represents the cobordism class of M, then M is cobordant to an M' which immerses in R^{m+k} if and only if x is in the image of λ_k. The essential point here is the use of the theorem of Hirsch [10] which reduces the question of immersion in R^{m+k} to the geometric dimension of the stable normal bundle of the manifold.

Stated in another way: we define an increasing filtration of $\pi_*(MO)$ by setting

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