

A FORMULA FOR RAMANUJAN'S τ -FUNCTION

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A recent investigation by Morris Newman shows that $\tau(p) \equiv 0 \pmod p$ for $p = 2, 3, 5, 7,$ and 2411 [1]. For his calculations Dr. Newman used a formula for $\tau(n)$ which involves $\sigma_3(n)$, the sum of the cubed divisors of n . Since a table of the exact values of $\sigma_3(n)$ was needed, his search for such p was limited to $p \leq 16,067$. This formula for $\tau(n)$ appears in [2] and a corrected version is given in [3]. In [2] there also appears a formula expressing $\sigma_3(n)$ in terms of $\sigma(n)$, the sum of the divisors of n . Thus it seemed possible that the ideas in [2] should lead to a formula for $\tau(n)$ in terms of $\sigma(n)$, which could then be used to extend Newman's search. This is indeed true and as is the case with most formulas in [2], once the formula is known, a simple proof of it can then be given. Using the new formula, Mr. K. Ferguson constructed a table of $\tau(p) \pmod p$ for $3 \leq p \leq 65,063$ which contained no new solutions of $\tau(p) \equiv 0 \pmod p$.

Since we have not found our formula in the literature and because it may be useful in other investigations, we state and prove it here.

THEOREM. *Let $e(z) = \exp(2\pi iz)$ and let $\tau(n)$ be defined by the equation $\sum_{n=1}^{\infty} \tau(n)e(nz) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24}$. Let $\sigma(n)$ be the sum of the positive divisors of n . Then for $n \geq 1$,*

$$\tau(n) = n^4 \sigma(n) - 24 \sum_{k=1}^{n-1} (35k^4 - 52k^3n + 18k^2n^2) \sigma(k) \sigma(n-k).$$

Proof. Let $\Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24}$ and put

$$f(z) = -\log \Delta(z) = -2\pi iz + 24 \sum_{n=1}^{\infty} \frac{\sigma(n)e(nz)}{n}.$$

As one easily sees, the theorem will be proved if the validity of the following equation is demonstrated:

$$2^9 \cdot 3\pi^6 \Delta(z) = 18[f'''(z)]^2 + f'(z)f^{(5)}(z) - 16f''(z)f^{(4)}(z).$$

Call the right-hand side of this equation $F(z)$. Then $F(z) = F(z+1)$ and the first Fourier coefficient of F is $2^9 \cdot 3\pi^6$. Hence, this equation will be proved if we show that $F(-1/z) = z^{12}F(z)$, since the space of cusp forms of weight 6 for the modular group is one-dimensional.

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