ISOMETRIES INDUCED BY COMPOSITION OPERATORS AND INVARIANT SUBSPACES[1]

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1. In this note, we consider some relations between some subspaces of $H^p(D)$ invariant under multiplication by z and some classes of isometries induced by linear fractional transformations mapping D onto D (l.f.t.). Here $D = \{|z| < 1\}$ and $H^p(D)$, $\infty > p \geq 1$, denotes the standard Hardy class of holomorphic functions. Given a l.f.t. ϕ, let C_ϕ and V_ϕ be defined on H^p by $C_\phi f = f \circ \phi$ and $V_\phi f = (\phi')^{1/p} C_\phi f$. (Note that the definition of V_ϕ depends on its domain H^p.) C_ϕ is a standard composition operator and is well known to be a bounded linear map of H^p onto H^p (see [5] for a discussion of composition operators). V_ϕ is clearly an isometry of H^p onto H^p, and further, F. Forelli has shown that for $p \neq 2$, every isometry of H^p onto H^p has the form bV_ϕ for some l.f.t. ϕ, where $b \in \mathbf{C}$, $|b| = 1$ [4]. We consider here the case where ϕ has a fixed point on $T = \{|z| = 1\}$, and for simplicity, we will assume $\phi(1) = 1$. Our main results are for H^2; in Theorem 1 we show that V_ϕ is a bilateral shift, and in Theorem 2 we show that a subcollection of $\{V_\phi\}$ generates a reflexive algebra which is related to a reflexive-type property of some other algebras.

2. For $c > 0$, $t \in \mathbf{R}$, let

$$\alpha_{c,t} = [t + i(c - 1)][t + i(c + 1)]^{-1},$$

and let $\phi_{c,t}(z) = (1 - \bar{\alpha})(1 - \alpha)^{-1}(z - \alpha)(1 - \bar{\alpha}z)^{-1}$ be the unique l.f.t. such that $\phi_{c,t}(\alpha_{c,t}) = 0$, $\phi_{c,t}(1) = 1$. Let $C_{c,t}$ and $V_{c,t}$ denote the corresponding maps induced by $\phi_{c,t}$, and for $r > 0$, let

$$\Delta_r(z) = \exp\left[-r(1 + z)(1 - z)^{-1}\right].$$

We note that by Beurling's theorem, $\{\Delta_r(z)H^p\}$ forms a decreasing family of invariant subspaces of H^p with $\bigcap_r \Delta_r H^p = \{0\}$.

LEMMA 1. *For $\alpha \in D$, there exists a unique $c > 0$, $t \in \mathbf{R}$ such that $\alpha = \alpha_{c,t}$. For $r > 0$, $V_{c,t}(\Delta_r H^p) = (\Delta_{rc^{-1}} H^p)$.*

Proof. Consider $\psi\colon \prod^+ \to D$ by $\psi(w) = (w - 1)(w + 1)^{-1}$, where $\prod^+ = \{\operatorname{Re} w > 0\}$. Then $\operatorname{Re} w = c$ iff $\psi(w) = \alpha_{c,t}$, and, in fact,

$$\alpha_{c,t} \in \{z \mid |z - c(c + 1)^{-1}| = (c + 1)^{-1}\},$$

the circle in D of radius $(c + 1)^{-1}$ tangent to T at 1. A direct computation

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