CUBIC FORMS IN GAUSSIAN VARIABLES

BY

F. ALBERTO GRÜNBAUM

Introduction

Let $X_1, \ldots, X_n$ be independent Gaussian variables with mean zero and variance one.

It is often of interest to determine the distribution of a polynomial in these variables, the simplest cases being the linear and the quadratic ones.

These two cases are fairly simple and can be best (for our purposes) summed up as follows. If $P(X_1, \ldots, X_n)$ and $Q(X_1, \ldots, X_n)$, in short $P(X)$ and $Q(X)$, have the same distribution, then there exists an orthogonal transformation O in R^n such that

$$P = Q_\circ O. \tag{1}$$

In this paper we study the homogeneous cubic case. Our results are complete only in the case of a pair of Gaussian variables and thus we limit ourselves to this special case. The same methods, though, serve to uncover some results both for the cubic case in an arbitrary number of variables, as well as in the case of an arbitrary polynomial in two variables.

Preliminaries

We are interested in polynomials of the type

$$P(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

and their distribution function when x and y are independent Gaussian variables with mean zero and variance one. This means that we are concerned with the function

$$\mu_P(\lambda) = \iint_{P_\lambda} e^{-(x^2+y^2)/2}\, dx\, dy \quad \text{where } P_\lambda = \{(x, y) \in R^2 \mid P(x, y) \leq \lambda\}.$$

The function $\mu_P(\lambda)$ is called the distribution of the polynomial P and can be defined in terms of the identity

$$\int_{R^2} e^{i\xi P(x,y)} e^{-(x^2+y^2)/2}\, dx\, dy = \int_{-\infty}^{\infty} e^{i\xi\lambda}\, d\mu_P(\lambda) \tag{2}$$

which holds for all real ξ.

For the cases when P is a linear or a quadratic polynomial (2) can be evaluated explicitly and one is led directly to conclude that there is a one-to-one relation

Received July 1, 1974.