HEIGHT FUNCTIONS ON SURFACES WITH THREE CRITICAL POINTS

BY

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We shall consider the following question: Under what conditions can we find an embedding or an immersion f: $M^2 \to E^3$ of a closed surface M^2 into Euclidean 3-space such that there is a linear function on E^3, z: $E^3 \to \mathbf{R}$, so that the composition zf: $M^2 \to \mathbf{R}$ has exactly three critical points, one of which may be degenerate. This question for a smooth embedding f was suggested as an exercise by H. Hopf in [6, p. 92]. The only possibility of a three critical point (3cp) smooth embedding is the case of a 2-sphere embedded as a "shoe surface" (see Figure 1).

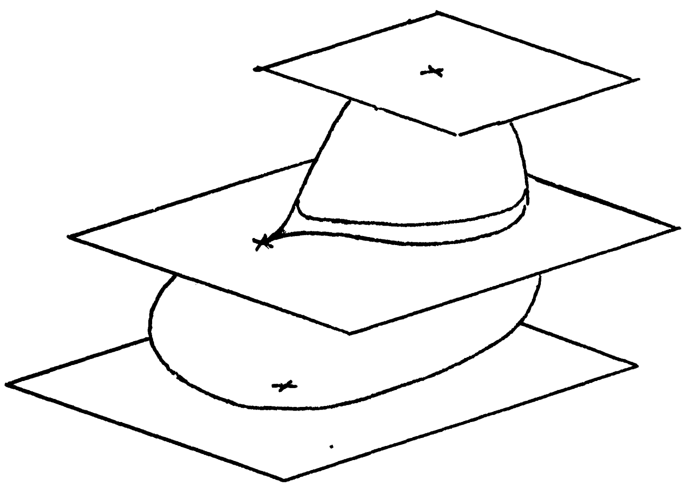

FIGURE 1

When we consider smooth immersions or polyhedral embeddings or immersions, the results are quite different. In this note, we prove the following results.

THEOREM 1. *There is a smooth* 3cp *immersion of the torus but there is no smooth* 3cp *immersion for any orientable surface of genus greater than one.*

THEOREM 2. *There is a smooth* 3cp *immersion of any nonorientable surface.*

These immersions can be approximated by polyhedral 3cp immersions but in the polyhedral case we have an entirely different phenomenon:

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