# EMBEDDING SPACES

BY

R. LASHOF

## 0. Introduction

In a recent paper [10], Helen Robinson gives an improvement of a theorem of Dax relating smooth and topological embeddings of manifolds in the metastable range. Their result is also a consequence of the techniques of Morlet [9]. In this paper we extend the results of Robinson using Morlet's idea of relating embeddings and immersions, and recent results of Millett on *PL*-immersions [8]. What we show is that in a range of dimensions above the metastable (Corollary 3 of Theorem A) the obstructions to deforming higher homotopy groups of topological embeddings to smooth embeddings lie in the Haefliger knot groups. We also relate topological and piecewise linear (*PL*) embeddings. In Section 2, we relate *PL* embeddings to the space of maps, extending a result of Lusk [6]. For a range of dimensions, this reduces the computation of the homotopy groups of spaces of topological, piecewise linear and smooth embeddings to a purely homotopy problem.

## 1. The relationship between smooth and topological embeddings

Let $(M^p, \partial M) \subset (N^n, \partial N)$ be smooth manifolds, $M$ compact (with possibly $\partial M = \emptyset$, $\partial N = \emptyset$). Let $E^t(M, N)$ (resp. $E^d(M, N)$) be the space of locally flat topological (resp. smooth) embeddings rel $\partial$. These may be treated as spaces with the *C-O* topology (resp. $C^\infty$-topology) or as $\Delta$-sets (see Appendix for a detailed discussion). $\mathrm{Im}^t\,(M, N)$ (resp. $\mathrm{Im}^d\,(M, N)$) will be the corresponding spaces of immersions rel $\partial$. Also Maps $(M, N)$ will be the space of continuous maps rel $\partial$. Let $T$ be a closed normal tube of $M$ in $N$, and $\mathring{T}$ an open normal tube containing $T$, defined with respect to some metric on $N$. Then $E^t(T, N)$ (resp. $E^d(T, N)$) will denote the space of locally flat (smooth) embeddings of $T$ in $N$ rel $T \cap \partial N$; and similarly for $E(T, \mathring{T})$. Finally, let $E(T, \mathring{T} \bmod M)$ be the subspace of $E(T, \mathring{T})$ of embeddings fixed on $M \cup (T \cap \partial N)$. We assume $n \geq 5$ throughout this paper.

By the isotopy extension theorem (see [2]) the restriction map $E(T, N) \to E(M, N)$ is a fibration (i.e., $E(T, N)$ is a fibre space over a union of components of $E(M, N)$) with fibre $E(T, N \bmod M)$. In either category, $E(T, \mathring{T} \bmod M)$ is a deformation retract of $E(T, N \bmod M)$. Thus (up to homotopy equivalence) the following are fibrations in both categories:

$$\text{(a)} \qquad \begin{aligned} &E(T, \mathring{T} \bmod M) \to E(T, N) \to E(M, N),\\ &E(T, \mathring{T} \bmod M) \to E(T, \mathring{T}) \to E(M, \mathring{T}). \end{aligned}$$

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