OBSTRUCTIONS TO LIFTING *-MORPHISMS INTO THE CALKIN ALGEBRA

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1. Introduction

Let H be a separable infinite dimensional Hilbert space, $\mathscr{B}(H)$ the algebra of bounded operators on H, $\mathscr{K}(H)$ the set of compact operators, $\mathscr{A}(H) = \mathscr{B}(H)/\mathscr{K}(H)$, $\pi\colon \mathscr{B}(H) \to \mathscr{A}(H)$ the quotient map. In their paper [1], Brown, Douglas, and Fillmore investigate for a compact metric space X the group Ext (X) consisting of unitary equivalence classes of unital injective *-morphisms $\tau\colon C(X) \to \mathscr{A}(H)$. This group completely solves (in principle at least) the lifting problem for injective unital *-morphisms τ from $C(X)$ to $\mathscr{A}(H)$: namely, there is a *-morphism $\tilde{\tau}$ which makes the diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{\tilde{\tau}} & \mathscr{B}(H) \\ & {\scriptstyle\tau}\searrow & \downarrow{\scriptstyle\pi} \\ & & \mathscr{A}(H) \end{array}$$

commutative iff the equivalence class $[\tau]$ of τ in Ext (X) is 0. The lifting problem is meaningful for any injective *-morphism from a C^*-algebra, although in the general case there is no functor around with the pleasant group properties of Ext. In the case A is UHF, we give in this paper an essentially complete answer. We follow throughout the terminology and conventions of Dixmier [3].

2. The semigroup $E(A)$

To solve the lifting problem we use a semigroup of *-morphisms from a C^*-algebra A to $\mathscr{A}(H)$ which is a fairly straightforward generalization of the semigroup Ext of [1]. We let $M(A; H)$ be the set of injective *-morphisms $\tau\colon A \to \mathscr{A}(H)$ and $\tilde{M}(A; H)$ the set of maps $\tau'\colon A \to \mathscr{B}(H)$ such that the composition $\pi\tau'$ is an injective *-morphism. Introduce on $\tilde{M}(A; H)$ the relation $\equiv$ of unitary equivalence modulo the compacts (i.e., $\tau \tilde{\equiv} \rho$ iff there is a unitary U such that $U\rho(x)U^* - \tau(x) \in \mathscr{K}(H)$ for all $x \in A$). On $M(A; H)$ we consider the corresponding relation $\equiv$; (i.e., $\rho \equiv \tau$ iff there is a unitary $U \in \mathscr{B}(H)$ such that $\pi(U)\rho(x)\pi(U^*) = \tau(x)$ for all $x \varepsilon A$.) The quotient sets $\tilde{M}(A; H)/\tilde{\equiv}$ and $M(A; H)/\equiv$ are naturally equivalent; we denote them by $E(A)$; denote the class of $\tilde{\tau} \in \tilde{M}(A; H)$ in $E(A)$ (resp. the class of $\tau \in M(A; H)$) by $[\tilde{\tau}]^\sim$ (resp. $[\tau]$). Observe $E(A)$ is a contravariant functor in the category of C^*-algebras

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