ASSEMBLING COMPACT RIEMANN SURFACES WITH GIVEN BOUNDARY CURVES AND BRANCH POINTS ON THE SPHERE

BY

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1. Introduction

The conformal structure of a Riemann surface is determined by any nonconstant meromorphic function on the surface. Every closed Riemann surface may be presented as an n-sheeted covering of the Gaussian sphere branched over w points a_i on the sphere [2, p. 47]. Over eighty years ago, A. Hurwitz [11] showed how to associate with such a covering a system of w permutations H_i on n symbols whose product $H_1 H_2 \cdots H_w$ is the identity and which generate a transitive group on the n symbols. Draw w rays α_i from the a_i to the common reference point ∞. Lifting the α_i to the surface decomposes it into a finite cell complex with n faces covering the same slit region $S^2 - \bigcup \alpha_i$ on the sphere. The $2w$ edges cover the interiors of the arcs α_i and n of the vertices, each of degree w, lie over ∞. The remaining vertices, $\hat{a}_{ij}$, of degrees δ_{ij}, $i = 1, \ldots, w$, $j = 1, \ldots, k_i$, lie over the a_i. The permutation H_i has k_i cycles of lengths δ_{ij}, each permuting the faces incident to $\hat{a}_{ij}$ in cyclic order. The Euler characteristic of the surface satisfies the so-called Hurwitz-Riemann relation $\chi = 2n - \mu$, where $\mu = nw - \sum k_i = \sum (\delta_{ij} - 1)$ is the branching number of the covering. Note that the covering projection is locally $\delta_{ij}:1$ in a deleted neighborhood of $\hat{a}_{ij}$. Conversely, given a system of Hurwitz permutations H_i for the points a_i, the surface may be reassembled by identifying edges of n polygonal cells of $2w$ sides each according to the information contained in the H_i.

A related but considerably more difficult problem is to develop an analogous combinatorial characterization of bordered Riemann surfaces. For present purposes, let a *compact Riemann surface* be a pair (M, F), where M is a compact, connected topological 2-manifold with border ∂M, and F a continuous map from M to S^2 which is locally $1:1$ at all but a finite number of so-called *critical points* lying in $M - \partial M$. For each critical point $p \in M$ there is a nonnegative integer $\mu(p)$ so that F is locally topologically equivalent to the complex power function $w = z^{\mu(p)+1}$. (We shall discuss our terminology and justify the invocation of Riemann's name in Section 5.) Extend μ to vanish at noncritical points of M. For $q \in S^2$, set $\mu(q) = \tilde{\mu}(F^{-1}(q))$, where $\tilde{\mu}$ is the obvious numerical measure on the subsets of M induced by μ. If ∂M is empty we say that M is a *closed*

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