EXTENSIONS OF ERGODIC GROUP ACTIONS

BY

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In this paper we shall study extensions in the theory of ergodic actions of a locally compact group. If G is a locally compact group, by an ergodic G-space we mean a Lebesgue space (X, μ) together with a Borel action of G on X, under which μ is invariant and ergodic. If (X, μ) and (Y, ν) are ergodic G-spaces, (X, μ) is called an extension of (Y, ν) and (Y, ν) a factor of (X, μ) if there is a Borel function $p: X \to Y$, commuting with the G-actions, such that $p_*(\mu) = \nu$. Various properties that one considers for a fixed ergodic G-space have as natural analogues properties of the triple (X, p, Y) in such a way as to reduce to the usual ones in case Y is a point. This is the idea of "relativizing" concepts, which is a popular theme in the study of extensions in topological dynamics. In ergodic theory, relativization is a natural idea from the point of view of Mackey's theory of virtual groups [16]. Although familiarity with virtual groups is not essential for a reading of this paper, this idea does provide motivation for some of the concepts introduced below, and a good framework for understanding our results. We shall therefore briefly review the notion of virtual group and indicate its relevance.

If X is an ergodic G-space, one of two mutually exclusive statements holds:

(i) There is an orbit whose complement is a null set. In this case, X is called essentially transitive.

(ii) Every orbit is a null set. X is then called properly ergodic.

In the first case, the action of G on X is essentially equivalent to the action defined by translation on G/H, where H is a closed subgroup of G; furthermore, this action is determined up to equivalence by the conjugacy class of H in G. In the second case, no such simple description of the action is available, but it is often useful to think of the action as being defined by a "virtual subgroup" of G. Many concepts defined for a subgroup H, can be expressed in terms of the action of G on G/H; frequently, this leads to a natural extension of the concept to the case of an arbitrary virtual subgroup, i.e., to the case of an ergodic G-action that is not necessarily essentially transitive. Perhaps the most fundamental notions that can be extended in this way are those of a homomorphism, and the concomitant ideas of kernel and range. These and other related matters are discussed in [16].

From this point of view, the notion of an extension of an ergodic G-space has a simple interpretation. A measure preserving G-map $\phi: X \to Y$ can be viewed as an embedding of the virtual subgroup defined by X into the virtual subgroup defined by Y. Thus, it is reasonable to hope that many of the concepts that one considers for a given ergodic G-space, i.e., a virtual subgroup of G,

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