ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. I

BY

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1. Introduction

Let E be an arbitrary nonempty set of (positive rational) prime numbers. For each positive integer n, let $\omega(n; E)$ denote the number of distinct primes in E which divide n, and let $\Omega(n; E)$ be the total number of primes in E which divide n, counted according to multiplicity. Thus if $n = p_1^{a_1} \cdots p_r^{a_r} n'$, where $r \geq 0$, $p_1, \ldots, p_r$ are distinct primes in E, each a_j is a positive integer, and n' has no prime factor in E, then $\omega(n; E) = r$ and $\Omega(n; E) = a_1 + \cdots + a_r$. When E is the set of all primes, we write $\omega(n; E) = \omega(n)$, $\Omega(n; E) = \Omega(n)$. The main objective of this paper and a subsequent one [31] is to derive some very accurate information about the distribution of values of these functions.

In the study of the sizes of $\omega(n; E)$ and $\Omega(n; E)$, an important role is played by the function

$$E(x) = \sum_{p \leq x,\, p \in E} p^{-1}. \tag{1.1}$$

For example, it is easy to show by the method of [20, Section 22.10] that $\omega(n; E)$ has average order $E(n)$, and the same is true of $\Omega(n; E)$ if $E(n)$ tends to infinity with n. Furthermore, a general theorem of Turán [38] shows in particular that if $E(x) \to +\infty$ as $x \to +\infty$, then each of $\omega(n; E)$ and $\Omega(n; E)$ has normal order $E(n)$. Turán's proof yields a quantitative version of this result which can be stated as follows for $\omega(n; E)$: if $\alpha = \alpha(x) > 0$ and $E(x) \geq 1$, then

$$\text{card}\,\{n\colon n \leq x \text{ and } |\omega(n; E) - E(x)| \geq \alpha E(x)\} \leq c_1 x/\alpha^2 E(x). \tag{1.2}$$

(Throughout this paper, card B denotes the number of members of the set B. For $i = 1, 2, \ldots, c_i(\delta, \varepsilon, \ldots)$ means a positive number depending only on $\delta, \varepsilon, \ldots$, while c_i means a positive absolute constant.) In particular,

$$\text{card}\,\{n\colon n \leq x \text{ and } |\omega(n; E) - E(x)| \geq \alpha E(x)\} = o(x) \tag{1.3}$$

if $E(x) \to +\infty$ and $\alpha E(x)^{1/2} \to +\infty$ as $x \to +\infty$. The result (1.3) was first proved by Hardy and Ramanujan [19] (this paper is reprinted in [33, pp. 262–275]) for the special case in which E is the set of all primes (in this case, it is well known that $E(x) = \log\log x + O(1)$ for $x \geq 2$, and in fact, $E(x)$ can be replaced by $\log\log x$ in (1.3)).

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