COMPLEMENTARY 1-ULC PROPERTIES FOR 3-SPHERES IN 4-SPACE

BY

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1. Introduction

A recurring theme in geometric topology is the importance of the 1-ULC property. For example, if Σ is an $(n-1)$-sphere topologically embedded in S^n ($n \neq 4$), then Σ is flat if and only if $S^n - \Sigma$ is 1-ULC. (See [2], [8], and [5].) If U is a component of $S^n - \Sigma$, it is natural to ask: For which sets $T \subset \Sigma$ is it true that $U \cup T$ is 1-ULC? (Of course, $T = \Sigma$ always works.) For $n = 3$, R. H. Bing has shown [3] that for some 0-dimensional $T \subset \Sigma$, $U \cup T$ is 1-ULC. For $n \geq 5$, Robert J. Daverman has found [6] a 1-dimensional $T \subset \Sigma$ such that $U \cup T$ is 1-ULC. (It is suspected that the dimension of Daverman's set cannot, in general, be lowered but no example is yet at hand.)

In Theorem 1 we extend Daverman's result to cover the case $n = 4$. Moreover, as constructed our 1-dimensional set T is easily seen to have embedding dimension at most 1 relative to Σ ("$\text{dem}_\Sigma T \leq 1$"), in the sense of [13] and [10]. We cannot hope to strengthen Daverman's high-dimensional result to obtain $\text{dem}_\Sigma T \leq 1$, when $n \geq 5$. For in Theorem 2 we observe that when $n \geq 6$, if T can be found with $\text{dem}_\Sigma T \leq 1$, then T can be chosen so that $\text{dem}_\Sigma T \leq 0$. But in [7], Daverman constructs embeddings of Σ in S^n, for all $n \geq 4$, for which T can never be chosen to have $\text{dem}_\Sigma T \leq 0$. In fact, in these examples T must satisfy $\text{dem}_\Sigma T \geq n - 3$.

We account for our inability to obtain $\text{dem}_\Sigma T \leq 1$ when $n > 4$ by remarking that for a σ-compactum T in Σ, "$\text{dem}_\Sigma T \leq 1$" is a stronger statement when $\dim \Sigma > 3$ than when $\dim \Sigma = 3$. For when $\dim \Sigma > 3$, $\text{dem}_\Sigma T \leq 1$ implies $\Sigma - T$ is 1-ULC. No such implication holds when $\dim \Sigma = 3$. We can appreciate the relative weakness of the statement "$\text{dem}_\Sigma T \leq 1$" when $\dim \Sigma = 3$ in another way: James W. Cannon has observed that when $\dim \Sigma = 3$, "$\text{dem}_\Sigma T \leq 1$" is equivalent to the existence of a 0-dimensional subset S of T such that $(\Sigma - T) \cup S$ is 1-ULC. However when $\dim \Sigma > 3$, any codimension 2 σ-compactum T in Σ contains a 0-dimensional subset S for which $(\Sigma - T) \cup S$ is 1-ULC.

Examples are easily constructed in all dimensions $n \geq 3$ with the property that any subset T of Σ for which $U \cup T$ is 1-ULC must be dense in Σ. Thus the subset T constructed by Daverman and the present authors is, in general, noncompact. In fact, Carl Pixley has noted that for $n \geq 5$, if T is a *compact*

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