THE KERNEL OF THE LOOP SUSPENSION MAP

BY

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Let X be a 1-connected H space such that the Hopf algebra $H^*(X; Z_p)$ has finite type. In this paper we characterize elements of the kernel of the loop map

$$\sigma\colon QH^q(X; Z_p) \to PH^{q-1}(\Omega X; Z_p)$$

both in terms of restricted types of Massey products and, of more interest, in terms of elementary stable cohomology operations. Basically the main result, Theorem B, states that if $\sigma x = 0$, then either $x \in \beta_k \mathscr{P}^I(u)$ or $x \in \beta_k \mathscr{P}^J \psi_r(v)$, where β_k is the p^kth order Bockstein, $\mathscr{P}^I$ and $\mathscr{P}^J$ are particularly simple primary operations, ψ_r is a specific secondary cohomology operation, and u and v are indecomposable cohomology classes of $H^*(X; Z_p)$. One of the applications is a characterization of differentials in certain spectral sequences in terms of these stable cohomology operations.

Section 1

Let $H_*(X)$ and $H^*(X)$ denote mod p singular homology and cohomology theories for a fixed prime p. If $\pi\colon PX \to X$ is the standard path space fibration with fiber the loop space ΩX, then the loop suspension map is the composite $\sigma = \delta^{-1}\pi^* j^{*-1}$:

$$H^q(X) \underset{\sim}{\xleftarrow{j^{*-1}}} H^q(X, x_0) \xrightarrow{\pi^*} H^q(PX, \Omega X) \underset{\sim}{\xleftarrow{\delta^{-1}}} \tilde{H}^{q-1}(\Omega X) \quad \text{for } q \geq 1.$$

This map was first introduced by Eilenberg and MacLane in their study of the relation between $K(\pi, n)$ and $K(\pi, n-1)$ [5]. It was generalized by Serre [23] to arbitrary fibrations. G. W. Whitehead [29] showed that σ annihilates decomposables of $H^*(X)$ and that $\operatorname{Im} \sigma \subset PH^*(\Omega X)$, the submodule of primitives in the Hopf algebra $H^*(\Omega X)$. Thus σ extends to a homomorphism:

$$\sigma\colon QH^q(X) \to PH^{q-1}(\Omega X).$$

The fact that σ annihilates decomposables was generalized to the statement $\sigma\langle u_1, \ldots, u_n\rangle = \{0\}$ for all Massey products [11]. Conversely J. P. May showed that every element of Ker σ belongs to some canonically defined matric Massey product (MMP) (see [19] or [8]).

If X is an H space, then $H^*(X)$ is a commutative Hopf algebra. Thus there are only a few multiplicative relations in $H^*(X)$. This means that there are only

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