COMPLETE SPACES AND TRI-QUOTIENT MAPS

BY

E. MICHAEL

1. Introduction

The purpose of this note is to study the behavior of complete spaces under various kinds of maps.[1] We first do this for open maps, where we give new proofs for some known results, and then extend these results to tri-quotient maps, an interesting new concept which is introduced and studied in this paper. As an application of our results, we generalize a recent theorem of J. P. R. Christensen [7] on compact-covering images of complete separable metric spaces.[2]

The most familiar completeness properties are complete metrizability and Čech-completeness,[3] and it is these concepts which will mainly concern us in this introduction. In the body of the paper, however, we shall mostly work with sieve-completeness, a particularly well-behaved property which was recently introduced in [5] by J. Chaber, M. M. Čoban, and K. Nagami (see Definition 2.1).[4] Every Čech-complete space is sieve-complete, and it was shown in [5] that the two concepts are equivalent in the presence of paracompactness. The proof of that equivalence in [5] was, however, quite indirect (see Remarks 8.1, 8.2, 8.3), and our first task will be to give a simple, direct proof of this result (see Theorem 3.2). That permits us to prove our basic mapping theorems for sieve-complete spaces, where they are particularly simple, and to indicate in this introduction some consequences of these results for Čech-complete spaces and completely metrizable spaces.

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[1] All maps in this paper are continuous surjections. No separation properties are assumed unless indicated; however, regular spaces are T_1 and paracompact spaces are Hausdorff.

[2] I would like to take this opportunity to thank Eric van Douwen for numerous helpful conversations and suggestions during the summer of 1975. In particular, this paper was originally motivated by his question of whether Christensen's theorem could be generalized to the result stated in Corollary 1.5, and it was he who distilled the concept of a tri-quotient map out of the author's original proof of that result.

[3] A completely regular space is *Čech-complete* if it is a G_δ in one (equivalently, in all) of its Hausdorff compactifications. By a result of E. Čech ([16] or [8, p. 190, Theorem 11]), a metrizable space is completely metrizable (i.e., metrizable by a complete metric) if and only if it is Čech-complete.

[4] Sieve-complete spaces are called *monotonically Čech-complete* in [5]. It follows from [5, Lemma 1.1 and Proposition 2.10] that a regular space is sieve-complete if and only if it is a λ_b-space in the sense of H. H. Wicke [28]. However, sieve-complete spaces seem to be much easier to work with than λ_b-spaces (even when the two are equivalent), and—unlike λ_b-spaces—many results about them are true without assuming any separation properties at all (see, for example, Proposition 4.2 and Theorem 6.3).