SUBGROUPS WITH TRIVIAL MAXIMAL INTERSECTION

BY

STEVEN BAUMAN

In a group G, let $\Phi(G)$ be the intersection of all maximal subgroups. If $H \leq G$, then it is clear that $H \leq \Phi(G)$ if and only if $H \leq M$ for every maximal subgroup M of G. It is well known that if G is finite then $\Phi(G)$ is a nilpotent group. It follows that if $H \cap M = H$ for all maximal subgroups M of a finite group G, then H is nilpotent. In this note we will consider a similar situation.

DEFINITION. A subgroup H of G is said to satisfy $\mathscr{P}(G)$ if for any maximal subgroup M of G either $H \cap M = H$ or $H \cap M = \langle 1 \rangle$.

It is proved in [1] that if G is finite and solvable then if H satisfies $\mathscr{P}(G)$, H is nilpotent. In this note we provide more information about H. In particular, we say something of the embedding of H in G when H satisfies $\mathscr{P}(G)$.

All groups will be finite and most notations standard. We use $M <\cdot\, G$ for M being a maximal subgroup of G.

LEMMA 1. *Let $K \leq H < G$ with H satisfying $\mathscr{P}(G)$. If $N \lhd G$ then K satisfies $\mathscr{P}(G)$ and HN/N satisfies $\mathscr{P}(G/N)$.*

Proof. The statement about K is clear. Let $M/N <\cdot\, G/N$. Then Dedekind's theorem yields

$$\frac{HN}{N} \cap \frac{M}{N} = \frac{(H \cap M)N}{N}.$$

Since H satisfies $\mathscr{P}(G)$ the result follows.

There are some particular situations where subgroups H satisfying $\mathscr{P}(G)$ arise. For example, if $H \leq \Phi(G)$ or $H \leq N$ where N is a minimal normal subgroup of a solvable group G, then H satisfies $\mathscr{P}(G)$. Let G be a Frobenius group with kernel N and complement M. If N is minimal normal in G and $H \leq \Phi(M)$, then H is easily seen to satisfy $\mathscr{P}(G)$. Thus Frobenius actions sometimes give rise to subgroups satisfying $\mathscr{P}(G)$.

DEFINITION. A group H is said to be of Frobenius type if it has Sylow p-subgroups which are cyclic for $p > 2$ and cyclic or generalized quaternion for $p = 2$.

LEMMA 2. *Let H satisfy $\mathscr{P}(G)$ in a solvable group G. If N is a minimal normal complemented subgroup of G with $(|H|, |N|) = 1$, then either*

(1) $[H, N] = 1$ *or*
(2) *H is of Frobenius type.*

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