EVERY PLANAR MAP IS FOUR COLORABLE PART II: REDUCIBILITY[1]

BY

K. APPEL, W. HAKEN, AND J. KOCH

1. Introduction

In Part I of this paper, a discharging procedure is defined which yields the unavoidability (in planar triangulations) of a set $\mathcal{U}$ of configurations of ring size fourteen or less. In this part, $\mathcal{U}$ is presented (as Table $\mathcal{U}$ consisting of Figures 1-63) together with a discussion of the reducibility proofs of its members.

When the term reducible is used above it is used in the following formal sense. Every configuration in $\mathcal{U}$ has the property that it is not only C- or D-reducible in the sense of [16], [27] (references are to the bibliography of Part I), but also if it is arbitrarily immersed in a planar map (i.e., not necessarily "properly embedded") then that planar map cannot be a minimal five chromatic map. A rather detailed study of such "immersion reducibility" is included in this paper.

Every configuration in $\mathcal{U}$ of ring size eleven or greater has been checked by our computer programs, with one exception.[2] For the reducibility of configurations of smaller ring size we rely on the tables in [2]. We do not claim to have been first to reduce all of these configurations. In particular we understand that F. Allaire has made a complete list of reducible eleven-rings and that H. Heesch has a large list of reducible configurations which has not been published. Furthermore, since we did not apply splicing arguments, there are C-reducible configurations, some of which appear in [25] and [1], for which we were not able to find reducers. But, since it meant only a small enlargement of our set $\mathcal{U}$ we preferred to include in $\mathcal{U}$ only such configurations as we could verify with our programs.[2] (See the note at the bottom of page 490.)

Received July 23, 1976.

[1] We should like to express our appreciation to the Research Board of the University of Illinois for supporting the computing effort. We have received tremendous help from the Computer Services Office (C.S.O.) at University of Illinois in using not only the IBM 360-75 computer at Urbana but also the IBM 370-158 computer at Chicago Circle and the 370-168 computer of the University Administrative Data Processing Unit. We should like to especially thank the consultants and systems programmers at C.S.O. for their excellent help and advice and the operations staff for their superb cooperation. We should also like to thank Laurel, Peter, and Andrew Appel for careful checking of diagrams and verifying the occurrence of configurations in the results of the discharging procedure.

In particular, we want to thank Michael Rolle, Charles Mills, and William Mills for pointing out copying errors in the earlier preprints of this paper.

[2] There is one major exception to our policy of reducing all required configurations of ring size greater than ten. Early in our work we realized that Configuration 16-14, which we could not reduce, would, if reducible, enable us to simplify our argument. We asked Frank