EVERY PLANAR MAP IS FOUR COLORABLE
PART II: REDUCIBILITY

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1. Introduction

In Part I of this paper, a discharging procedure is defined which yields the
unavoidability (in planar triangulations) of a set $\mathcal{U}$ of configurations of ring
size fourteen or less. In this part, $\mathcal{U}$ is presented (as Table $\mathcal{U}$ consisting of
Figures 1-63) together with a discussion of the reducibility proofs of its members.

When the term reducible is used above it is used in the following formal
sense. Every configuration in $\mathcal{U}$ has the property that it is not only C- or D-
reducible in the sense of [16], [27] (references are to the bibliography of Part I),
but also if it is arbitrarily immersed in a planar map (i.e., not necessarily “properly
embedded”) then that planar map cannot be a minimal five chromatic
map. A rather detailed study of such “immersion reducibility” is included in
this paper.

Every configuration in $\mathcal{U}$ of ring size eleven or greater has been checked by
our computer programs, with one exception. For the reducibility of con-
figurations of smaller ring size we rely on the tables in [2]. We do not claim to
have been first to reduce all of these configurations. In particular we under-
stand that F. Allaire has made a complete list of reducible eleven-rings and that
H. Heesch has a large list of reducible configurations which has not been pub-
lished. Furthermore, since we did not apply splicing arguments, there are C-
reducible configurations, some of which appear in [25] and [1], for which we
were not able to find reducers. But, since it meant only a small enlargement of
our set $\mathcal{U}$ we preferred to include in $\mathcal{U}$ only such configurations as we could
verify with our programs.

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configurations in the results of the discharging procedure.

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out copying errors in the earlier preprints of this paper.

There is one major exception to our policy of reducing all required configurations of ring
size greater than ten. Early in our work we realized that Configuration 16-14, which we
could not reduce, would, if reducible, enable us to simplify our argument. We asked Frank