EVERY PLANAR MAP IS FOUR COLORABLE PART I: DISCHARGING[1]

BY

K. APPEL AND W. HAKEN

1. Introduction

We begin by describing, in chronological order, the earlier results which led to the work of this paper. The proof of the Four Color Theorem requires the results of Sections 2 and 3 of this paper and the reducibility results of Part II. Sections 4 and 5 will be devoted to an attempt to explain the difficulties of the Four Color Problem and the unusual nature of the proof.

The first published attempt to prove the Four Color Theorem was made by A. B. Kempe [19] in 1879. Kempe proved that the problem can be restricted to the consideration of "normal planar maps" in which all faces are simply connected polygons, precisely three of which meet at each node. For such maps, he derived from Euler's formula, the equation

$$4p_2 + 3p_3 + 2p_4 + p_5 = \sum_{k=7}^{k_{\max}} (k - 6)p_k + 12 \tag{1.1}$$

where p_i is the number of polygons with precisely i neighbors and $k_{\max}$ is the largest value of i which occurs in the map. This equation immediately implies that every normal planar map contains polygons with fewer than six neighbors.

In order to prove the Four Color Theorem by induction on the number p of polygons in the map ($p = \sum p_i$), Kempe assumed that every normal planar map with $p \leq r$ is four colorable and considered a normal planar map M_{r+1} with $r + 1$ polygons. He distinguished the four cases that M_{r+1} contained a polygon P_2 with two neighbors, or a triangle P_3, or a quadrilateral P_4, or a pentagon P_5; at least one of these cases must apply by (1.1). In each case he

Received July 23, 1976.

[1] The authors wish to express their gratitude to the Research Board of the University of Illinois for the generous allowance of computer time for the work on the discharging algorithm. They also wish to thank the Computer Services Organization of the University of Illinois and especially its systems consulting staff for considerable technical assistance. They further wish to thank Armin and Dorothea Haken for their effective assistance in checking the definitions and diagrams in the manuscript.

Haken also wishes to thank the Center for Advanced Study of the University of Illinois for support for the year 1974–75 and the National Science Foundation for support for half of the year 1971–72 and for summers 1971 through 1974. He also wishes to thank his teacher, Karl-Heinrich Weise at the University of Kiel, for introducing him to mathematics and in particular to the Four Color Problem.

Appel wishes to thank his teacher, Roger Lyndon, for teaching him how to think about mathematics.