ILLINOIS JOURNAL OF MATHEMATICS
Volume 21, Number 4, December 1977

MULTIPLIER SEQUENCES FOR FIELDS

BY

THOMAS CRAVEN AND GEORGE CSORDAS

1. Introduction

Let F be a field and $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of elements in F. If for every polynomial

$$f(x) = \sum_{k=0}^{n} a_k x^k, \quad a_k \in F,$$

which splits over F, the polynomial $\Gamma[f(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ also splits over F, then Γ is called a *multiplier sequence* for F. In the case when F is the field $\mathbf{R}$ of real numbers this concept was first introduced in 1914 by Pólya and Schur in their celebrated paper [9] entitled *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen.* This beautiful paper has been the fountainhead of numerous later investigations. The main result of this work has been hailed by R. P. Boas [1, p. 418] as a "key result on the boundary between Algebra and Analysis." Pólya and Schur have shown that all the multiplier sequences for $\mathbf{R}$ are generated by entire functions which can be uniformly approximated in a neighborhood of zero by polynomials with only real (negative) zeros. (For a precise statement of this result see Section 3.) In subsequent developments, these entire functions found important applications in other fields: for example, in the theory of integral transforms [3], approximation theory [11], the theory of total positivity [4] and probability theory [5].

In this paper, inspired by the work of Pólya and Schur, we investigate and characterize the multiplier sequences for more general fields. In Section 2 we describe some of the intrinsic properties of multiplier sequences and establish the main techniques for analyzing multiplier sequences. Our results primarily concern the algebraic and arithmetic properties a field must have in order to possess a multiplier sequence of a prescribed form. In Section 3 we show that several properties of multiplier sequences for $\mathbf{R}$ are also enjoyed by multiplier sequences of an arbitrary ordered field. Moreover, with the aid of a theorem of Tarski, we are able to provide a particularly useful necessary and sufficient condition for a sequence to be a multiplier sequence for a real closed field (and for certain somewhat more general fields). In fact, it is shown that $\Gamma[f(x)]$ splits for all polynomials $f(x)$ which split and have degree less than or equal to n if and only if $\Gamma[(x+1)^n]$ splits and has all its roots of the same sign. Section 4 is devoted to the complete characterization of multiplier sequences for all finite fields. In the final section we provide a list of open questions.

Received March 29, 1976.