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# COMMUTING SUBNORMAL OPERATORS

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If $M$ and $N$ are commuting normal operators on a Hilbert space $\mathscr{K}$, if $\mathscr{H}$ is a subspace of $\mathscr{K}$ which is invariant for $M$ and $N$, and if $S$ and $T$ are the restrictions to $\mathscr{H}$ of $M$ and $N$ respectively, then $S$ and $T$ are commuting subnormal operators with commuting normal extensions. A recent example of Lubin shows that commuting subnormal operators need not have commuting normal extensions [6]. However, commuting subnormal operators $S$ and $T$ have commuting normal extensions if either $S$ or $T$ is normal [2, Theorem 8], if either $S$ or $T$ is cyclic [9, Theorem 3], or if either $S$ or $T$ is an isometry [8, Theorem 1].

In this paper it is shown that two commuting subnormal operators $S$ and $T$ have commuting normal extensions if the spectrum of $T$ is finitely connected and the spectrum of the minimal normal extension of $T$ is contained in the boundary of the spectrum of $T$. This generalizes a result of Slocinski who proves the theorem under the additional hypotheses that $T$ is pure and that the spectrum of $T$ does not divide the plane [8, Theorem 5]. In addition, an example is presented of two commuting subnormal operators without commuting normal extensions. This example is different from the aforementioned example of Lubin and perhaps more elementary.

The main theorem is proved in Section 1, the example is presented in Section 2, and two problems are stated in Section 3. In this paper, all Hilbert spaces are complex and separable, all subspaces are closed, and all operators are bounded.

## 1. A theorem on the existence of commuting normal extensions

The main theorem is a consequence of seven known theorems and two elementary facts. These nine results are recorded as Lemmas 1 through 9 below and the main theorem follows. Let $X$ be a compact subset of the complex plane, let $\chi$ be the function $\chi(z) = z$, let $\int \oplus \mathscr{H}_x \, d\mu(x)$ be a direct integral over $X$, and let $M_\chi$ on $\int \oplus \mathscr{H}_x \, d\mu(x)$ be the operator defined by the equation $M_\chi(f) = \chi f$. An operator $A$ on $\int \oplus \mathscr{H}_x \, d\mu(x)$ is said to be decomposable if for each $x$ in $X$ there is an operator $A_x$ on $\mathscr{H}_x$ such that the function $x \to \|A_x\|$ is bounded and Borel measurable on $X$ and

$$A(f)(x) = A_x(f(x)) \quad d\mu\text{-a.e.}$$

for all $f$ in $\int \oplus \mathscr{H}_x \, d\mu(x)$. The operator $A$ is denoted $\int \oplus A_x \, d\mu(x)$. The first two lemmas are proved in Dixmier [3, p. 208 and p. 164] and the third lemma is due to Bastian [1, Theorem 4.4].

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