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A BANACH SPACE NOT CONTAINING l_1 WHOSE DUAL BALL IS NOT WEAK* SEQUENTIALLY COMPACT

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The structure of Banach spaces with nonweak* sequentially compact dual balls was studied in [7], where it was proved that if X is separable and the unit ball of X^{**} is not weak* sequentially compact, then X^* contains a subspace isomorphic to $l_1(\Gamma)$ for some uncountable set Γ. Subsequently it was proved in [1] that if the unit ball of X^* is not weak* sequentially compact, then (a) either c_0 is a quotient of X or l_1 is isomorphic to a subspace of X, and (b) X has a separable subspace with nonseparable dual. In this note we give an example of a Banach space X whose dual ball is not weak* sequentially compact, but where X contains no subspace isomorphic to l_1. This answers a question posed by H. P. Rosenthal [7].

The example we construct draws on two ideas. First R. Haydon [2] exhibited a compact Hausdorff space K which is not sequentially compact such that $C(K)$ does not contain a subspace isomorphic to $l_1(\Gamma)$ for any uncountable set Γ. Central to this construction (and to ours) is the existence of a "thin" family of subsets of the integers which infinitely separates every infinite subset of the integers (see Lemma 1 below). Secondly, the space X we exhibit must be nonseparable. A key part of our construction is a nonseparable analogue of JT, the James tree (cf. [3] or [4]). This space has the property that JT^* is not separable, yet JT contains no isomorph of l_1. We recall the definition of JT below during the proof of Lemma 2.

Notation. If X is a Banach space and $(g_\alpha)_{\alpha \in I} \subseteq X$, then by $\langle g_\alpha \rangle_{\alpha \in I}$ we mean the linear span of the set $(g_\alpha)_{\alpha \in I}$, while $[g_\alpha]_{\alpha \in I}$ denotes the closure of $\langle g_\alpha \rangle_{\alpha \in I}$. Also if L and M are subsets of N, the set of natural numbers, then $|L|$ denotes the cardinality of L. $L \subset^a M$ means $|L \backslash M| < \infty$ and $L \cap M =^a \emptyset$ means $|L \cap M| < \infty$.

Other Banach space notation we use is standard and may be found in [5].

The definition of X.

LEMMA 1. *There is a well ordered set I, $<$ and a collection of infinite subsets of N, $(M_\alpha)_{\alpha \in I}$, such that:*

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