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SPLINE SPACES ARE OPTIMAL FOR L^2 n-WIDTH

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1. Introduction

Let $X = (X, \|\cdot\|)$ be a normed linear space, $\mathscr{K}$ a subset of X and X_n an n-dimensional linear subspace of X. The Kolmogorov n-width of $\mathscr{K}$ relative to X is defined by

$$d_n(\mathscr{K}; X) = d_n(\mathscr{K}) = \inf_{X_n} \sup_{x \in \mathscr{K}} \inf_{y \in X_n} \|x - y\|.$$

X_n is called an optimal subspace for $\mathscr{K}$ provided that

$$d_n(\mathscr{K}) = \sup_{x \in \mathscr{K}} \inf_{y \in X_n} \|x - y\|.$$

This concept of n-width was introduced by Kolmogorov in [8] and in his paper he finds the exact value of the n-width for

$$W^{2,r}[0, 1] = \{f: f^{(r-1)} \text{ abs. cont. on } (0, 1), \|f^{(r)}\| \leq 1\}$$
$$(\|\cdot\| = L^2 \text{ norm on } [0, 1]).$$

Roughly speaking Kolmogorov showed that the n-width corresponds to the nth eigenvalue of a boundary value problem and an optimal subspace is spanned by the first n eigenfunctions. Kolmogorov claimed that $W^{2,r}[0, 1]$ has a unique optimal subspace and as late as Tihomirov [13] this error was overlooked. It was first observed to be false by Karlovitz in [4] while in Ioffe and Tihomirov [2] it is conjectured that $W^{2,r}[0, 1]$ has an optimal spline subspace.

Subsequently, Karlovitz [5] explored the question of nonuniqueness of optimal subspaces in a general setting. The related question for min max and max min characterization of eigenvalues has been treated in Weinstein and Stenger's book [17].

A main goal of this paper is to prove that $W^{2,r}$ admits, for all r, optimal spline subspaces. There are in fact two; one of degree $r - 1$ and another of degree $2r - 1$.

Before stating exactly our result for $W^{2,r}$ we wish to point out that an effort

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