ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 2, June 1979

ON THE DERIVATIVE OF A POLYNOMIAL

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1. Introduction and statement of results

It is well known that if $p_n(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree at most n, then (for references see [16])

$$\max_{|z|=1} |p_n'(z)| \le n \max_{|z|=1} |p_n(z)|, \tag{1}$$

where equality holds if and only if $p_n(z)$ is a constant multiple of z^n. If $p_n(z)\neq 0$ in $|z|<1$, then [11], [5], [2]

$$\max_{|z|=1} |p_n'(z)| \le \frac{n}{2} \max_{|z|=1} |p_n(z)|. \tag{2}$$

On the other hand, we have [18]

$$\max_{|z|=1} |p_n'(z)| \ge \frac{n}{2} \max_{|z|=1} |p_n(z)| \tag{3}$$

if $p_n(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$. Hence in (2) (as well as in (3)) equality holds for all polynomials $p_n(z)$ of degree n which have all their zeros on $|z|=1$.

Inequality (2) can be replaced [12], [9] by

$$\max_{|z|=1} |p_n'(z)| \le \frac{n}{1+K} \max_{|z|=1} |p_n(z)| \tag{4}$$

if $p_n(z)\neq 0$ in $|z|<K$, where $K>1$. Here, we have equality if

$$p_n(z)=c_0\left\{1+\binom{n}{1}\frac{1}{K}ze^{i\alpha}+\cdots+\binom{n}{\nu}\frac{1}{K^\nu}(ze^{i\alpha})^n+\cdots+\frac{1}{K^n}(ze^{i\alpha})^n\right\}. \tag{5}$$

Besides, it can be shown that if a polynomial $p_n(z)$ of degree n having all its zeros in $|z|\ge K>1$ is not of this form, then strict inequality holds in (4). In other words, there is equality in (4) for $p_n(z)=\sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ in $|z|<K$ $(K>1)$ if and only if $|c_1/c_0|=n/K$.

Now let us consider the following problem. Given that the polynomial

$$f_n(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$$

is univalent in $|z|<1$ how large can $(\max_{|z|=1}|f_n'(z)|)/\max_{|z|=1}|f_n(z)|$ be? We may apply (4) to the polynomial $p_{n-1}(z)=f_n(z)/z=:\sum_{\nu=0}^{n-1} c_\nu z^\nu$ which is of

Received November 29, 1977.