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FUNCTIONS WHICH FOLLOW INNER FUNCTIONS

BY
KENNETH STEPHENSON[1]

If $\mathcal{U}$ denotes the unit disc and $A \subset \mathcal{U}$ has (logarithmic) capacity zero, Frostman proved that the universal covering map ϕ_A: $\mathcal{U} \to \mathcal{U} \setminus A$ is an inner function. ϕ_A has some very nice properties: On the one hand, any analytic function f mapping $\mathcal{U}$ into $\mathcal{U} \setminus A$ is subordinate to ϕ_A, that is, there exists an analytic function g on $\mathcal{U}$ so that $f = \phi_A \circ g$. On the other hand, there is a group Γ_A of Möbius transformations of $\mathcal{U}$ such that any function f which is automorphic with respect to Γ_A can be realized as a composition with ϕ_A, that is, there exists a function g on $\mathcal{U} \setminus A$ so that $f = g \circ \phi_A$. In this paper we exploit these properties to obtain results about the composition operators $C_\phi: f \to f \circ \phi$ for general inner functions ϕ.

We consider these operators on the Hardy spaces, the Smirnov class, and the space of meromorphic functions of bounded characteristic. X will denote any one of these spaces. An obvious *necessary* condition for a function f in X to be in the range of C_ϕ is that $f(\alpha) = f(\beta)$ whenever $\phi(\alpha) = \phi(\beta)$. We describe this property by saying that f *follows* ϕ. We prove in Section 3 that this surprisingly simple condition is also *sufficient*. In Section 4 we use this condition to characterize the range of C_ϕ as a linear submanifold of X, extending and simplifying a characterization due to Ball [4]. We conclude with comments and questions in Section 5.

The composition operators C_ϕ have been much studied, particularly in relation to Toeplitz operators. For work in this direction, see Abrahamse [1], Abrahamse and Ball [2], Ball [4], Nordgren [7], [8], and Thomson [12], [13]. Certain properties of the inner functions ϕ_A were studied by the author in [10]. We assume that the reader is familiar with the basic theory of functions of bounded characteristic, the notion of logarithmic capacity for plane sets, and the elementary properties of universal covering surfaces for plane regions. Appropriate references would be Duren [5], Tsuji [14], and Ahlfors and Sario [3], respectively.

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1. Preliminaries

All functions have domain $\mathcal{U}$, the unit disc, unless stated otherwise.

1.1. *Factorization.* H^∞ denotes the space of bounded analytic functions.

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