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A RUDIN-SHAPIRO TYPE THEOREM

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0. Introduction

Let $P_n(t)=\sum_{r=1}^{n} a_r \exp ik(r)t$ be a trigonometric polynomial with $\sum_{r=1}^{n} |a_r|=1$. General considerations show that $\sup|P_n(t)|\geq n^{-1/2}$. Direct constructions, of which the most powerful is due to Rudin and Shapiro show that we can find a constant B and polynomials $P_1, P_2, \ldots$ with $\sup|P_n(t)|\leq Bn^{-1/2}$. On the other hand the standard probabilistic constructions only give the existence of polynomials P_n with $\sup|P_n(t)|\leq Bn^{-1/2}(\log n)^{1/2}$ for some.

The natural extension to general locally compact Abelian groups G is to ask what we can say about $\sup_{\chi\in\hat{G}}|\hat{\mu}_n(\chi)|$ if μ is a measure on G with $\|\mu\|=1$ and supp μ consisting of n points or less. In general we cannot say much, unless G is a finite Abelian group and we have bounds on the number of elements of G. This problem was investigated by Varopoulos and by Kaufman. In the first section we give an exposition of Kaufman's elegant probabilistic method and show that it gives considerably better results than the author claims. Typically, we can show that there exists a constant B such that for each $n\geq 1$ we can find $a_1, a_2, \ldots, a_n \in \mathbf{R}$ with

$$\sum_{r=1}^{n} |a_r|=1 \quad \text{and} \quad \left|\sum_{r=1}^{n} a_r\omega^r\right| \leq Bn^{-1/2}(\log n)^{1/2}$$

whenever ω is an n-th root of unity.

We give a very detailed description of this method, since in Section 3 we modify it to obtain, again by probabilistic means, improved results in which the $(\log n)^{1/2}$ factor is removed. Thus we can find a constant B such that for each $n\geq 1$ we can find $a_1, a_2, \ldots, a_n \in \mathbf{R}$ with

$$\sum_{r=1}^{n} |a_r|=1 \quad \text{and} \quad \left|\sum_{r=1}^{n} a_r\omega^r\right| \leq Bn^{-1/2}$$

whenever ω is an n-th root of unity. It is easy to deduce (as we do at the end of Section 2) the result stated in the last but one sentence of the first paragraph.

That this result can indeed be obtained by probabilistic means is perhaps the main point of interest for the general reader, but my purpose in writing

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