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MEASURABLE SUBBUNDLES IN LINEAR SKEW-PRODUCT FLOWS

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Introduction

The main purpose of this paper is to construct and discuss two ODEs and one real flow. Our work is based on techniques of topological dynamics developed by Furstenberg in [5]. These techniques also allow a discussion of certain other topics, namely Bohr's theorem, and a type of quasi-periodic function considered by Sell in [16] (see also Brezin, Ellis, and Shapiro [2]).

To avoid confusion due to the number of topics considered, we give here an outline of the paper, as well as an indication of why one might be interested in the ODEs and the flow. In §1, we prove Bohr's theorem using Furstenberg's techniques. In §2, we consider a certain irrational (Kronecker) flow $(K^2, \mathbf{R})$ on the 2-torus K^2. Let $\omega \cdot t$ represent the position of $\omega \in K^2$ after time t under this flow. We construct a non-continuous function $R \in L^2(K^2, m)$ (m is Lebesgue measure on K^2) and an analytic function b on K^2 such that "R is an antiderivative of b along orbits"; i.e.,

$$\int_0^t b(\omega \cdot s)\, ds = R(\omega \cdot \omega t) - R(\omega) \quad (\omega \in K^2, t \in \mathbf{R}).$$

The function b has mean value zero, but $\int_0^t b(\omega \cdot s)\, ds$ is not almost-periodic (a.p.). the functions R and b are of fundamental importance in constructing our three examples. In §2, we also consider Sell's results.

In §§ 3, 4, and 5, we treat the examples.

(§3) Consider the analytic differential equations E_ω:

$$\dot{x} = \frac{1}{2}\begin{bmatrix} 0 & -b(\omega \cdot t) \\ b(\omega \cdot t) & 0 \end{bmatrix} x \quad (\omega \in K^2, x \in \mathbf{R}^2).$$

We view this collection of ODEs as "generated" by some one quasiperiodic ODE E_{ω_0} (ω_0 a fixed element of K^2). The equations E_ω induce a "linear skew-product flow", or LSPF [13], [14], [15] on $K^2 \times \mathbf{R}^2$. This LSPF has interesting structure: the vector bundle $K^2 \times \mathbf{R}^2$ foliates into measurable, non-continuous, invariant, one-dimensional subbundles (3.3). Now, by Floquet theory, non-continuous subbundles cannot occur for periodic ODE's (3.4). The point we wish to make with this example is that, if one is to

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