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LACUNARY SPHERICAL MEANS

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0. Introduction and statement of results

Professor E. M. Stein introduced in [4] (see also [6]) the maximal function

$$S(f)(x) = \sup_{\varepsilon>0} \left| \int_{\Sigma} f(x-\varepsilon\alpha)\, d\sigma \right| \tag{0.1}$$

where f is any Borel measurable function defined on R^n, α is a point on the unit sphere Σ of R^n and $d\sigma$ stands for its "area" element. In the above paper Professor Stein proves the following result: If $n \geq 3$ and $p > n/(n-1)$, then

$$\|S(f)\|_p < C_p \|f\|_p. \tag{0.2}$$

If $p \leq n/(n-1)$ and $n \geq 2$ the result is false; what happens for $n = 2$ and $p > 2$ remains an open problem. Throughout this paper, we shall be concerned with the lacunary version of Stein's theorem. Define

$$\sigma(f)(x) = \sup_{k>0} \left| \int_{\Sigma} f(x-2^{-k}\alpha)\, d\sigma \right| \tag{0.3}$$

where k takes all the natural values. We have the following result:

0.4. THEOREM. *If $n \geq 2$, $p > 1$ and f is Borel measurable in R^n then*

(i) $$\|\sigma(f)\|_p < C_p \|f\|_p, \quad p > 1.$$

Moreover, we have the following inequality "near" L^1: If Q is a cube in R^n and $\lambda > 1/|Q|$ then

(ii) $$|Q \cap E(\sigma(f) > \lambda)| < \frac{C_1}{\lambda} |Q| + C_2 \frac{|\log \lambda|}{\lambda} \int_{R^n} |f| \, [1 + (\log^+ |f|) \log^+ \log^+ |f|] \, dx.$$

The constants C_1 and C_2 depend on n and Q but not on λ or f.

In particular, (ii) implies differentiability a.e. by lacunary spherical means in the Orlicz Class $L(\log^+ L) \log^+ \log^+ L$. Professor S. Wainger communicated to me that part (i) of the above theorem has been obtained also by R. R. Coifman and G. Weiss.

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