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MAXIMAL CHAINS OF PRIME IDEALS IN INTEGRAL EXTENSION DOMAINS, III

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1. Introduction

All rings in this paper are assumed to be commutative with identity, and the terminology is, in general, the same as that in [2].

To briefly describe the results in this paper, let A' denote the integral closure of an integral domain A in its quotient field, and let $s(A)$ (resp., $c(A)$) denote the set of lengths of maximal chains of prime ideals in A (resp., in arbitrary integral extension domains of A). Also, when $P \in \operatorname{Spec} A$, let $s(P)$ and $c(P)$ denote $s(A_P)$ and $c(A_P)$. Finally, let $\mathscr{C}$ denote the class of quasi-local domains R such that $c(R) = s(R)$. ($\mathscr{C}$ is an important class, since it contains all local domains occurring in algebraic and analytic geometry and in number theory. (In fact, all these rings satisfy the more stringent condition $c(R) = \{\text{altitude } R\}$.) Also, by [5, (4.1)] it contains all Henselian local domains and all local domains of the form $R[X]_{(M,X)}$, where (R, M) is an arbitrary local domain and X is an indeterminate. On the other hand, [2, Example 2, pp. 203–205] in the case $m = 0$ shows that not all local domains are in $\mathscr{C}$—but it has been conjectured, the Upper Conjecture (3.4), that this is essentially the only type of local domain not in $\mathscr{C}$.)

Our first theorem, (2.2), shows that if A is any integral domain and $P \in \operatorname{Spec} A$ is such that $c(P') = c(P)$, for all $P' \in \operatorname{Spec} A'$ that lie over P, then $c(Q) = c(P)$ whenever $Q \in \operatorname{Spec} B$ lies over P and B is an integral extension domain of A. We then show in (2.4) that every semi-local domain R has finite integral extension domains B such that all maximal ideals N in all integral extension domains C of B satisfy $C_N \in \mathscr{C}$ and $c(N) = c(M')$, for some maximal ideal M' in R'. Finally, in Section 3 we consider a new conjecture related to the results in Section 2, and show that it lies (implicationwise) between two previously studied chain conjectures.

I am indebted to the referee for a number of suggestions on improving this paper, and also for a considerable strengthening of my original versions of (2.1) and (2.2).

2. Two theorems on $c(P)$

In this section we prove two theorems concerning the behavior of $c(P)$, where P is a prime ideal in an integral domain A. To prove the first of these,

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