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SUBNORMAL OPERATORS AND HYPERINVARIANT SUBSPACES

BY

H. W. KIM AND CARL PEARCY

1. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator S in $\mathcal{L}(\mathcal{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator N in $\mathcal{L}(\mathcal{K})$ such that $N\mathcal{H} \subset \mathcal{H}$ and $N \mid \mathcal{H} = S$. (In this situation we say that N is a *normal extension* of S and that S is a *restriction* of N. Alternate characterizations of subnormal operators were given by Halmos [5] and Bram [2].) The operator N is called a *minimal* normal extension of S if the only reducing subspace for N containing $\mathcal{H}$ is $\mathcal{K}$ itself. It is well known that every subnormal operator has a minimal normal extension and that the minimal normal extension is unique up to unitary equivalence (cf. [5] or [7, p. 101]). Since subnormal operators are intimately related to their minimal normal extensions, and the spectral theorem guarantees the existence of a generous supply of invariant and hyperinvariant subspaces for (nonscalar) normal operators, the question whether every (nonscalar) subnormal operator in $\mathcal{L}(\mathcal{H})$ has a nontrivial invariant or hyperinvariant subspace has long been of interest, and remains open as of this writing. The purpose of this note is to make a modest contribution to this problem. We consider subnormal operators whose spectra have empty interior, and reduce the invariant subspace problem for this class of operators to a rather curious looking special case. This assumption on the spectrum is unpleasant, but a typical subnormal operator in this class has for spectrum a "Swiss cheese" with positive planar Lebesgue measure, and it is generally conceded that this class of subnormal operators is the most intractable with respect to the existence of invariant subspaces.

2. In what follows, the spectrum of an operator T will be denoted by $\sigma(T)$ and the essential spectrum (i.e., Calkin spectrum) of T by $\sigma_e(T)$. Let S be a nonscalar subnormal operator in $\mathcal{L}(\mathcal{H})$, and let N be a minimal normal extension of S acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$. If $\mathcal{K} = \mathcal{H}$, then S is normal and thus has nontrivial hyperinvariant subspaces, so we may assume that $\mathcal{K} \neq \mathcal{H}$ (which implies that S is not normal). We write $\mathcal{K} = \mathcal{H} \oplus (\mathcal{K} \ominus \mathcal{H})$ and note that it follows easily from the minimality of N that $\mathcal{K} \ominus \mathcal{H}$ has dimension $\aleph_0$ (i.e., $\mathcal{K} \ominus \mathcal{H}$ is neither finite dimensional nor nonseparable). We summarize these remarks as follows.

PROPOSITION 2.1. *Let S be a nonnormal subnormal operator in $\mathcal{L}(\mathcal{H})$. Then its minimal normal extension N may be taken to act on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, and*

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